Two-Way Source Coding With a Helper

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Abstract—Consider the two-way rate-distortion problem in which a helper sends a common limited-rate message to both users based on side information at its disposal. We characterize the region of achievable rates and distortions when the Markov relation (Helper)-(User 1)–(User 2) holds. The main insight of the result is that in order to achieve the optimal rate, the helper may use a binning scheme, as in Wyner–Ziv, where the side information at the decoder is the "further" user, namely, User 2. We derive these regions explicitly for the Gaussian sources with square error distortion, analyze a tradeoff between the rate from the helper and the rate from the source, and examine a special case where the helper has the freedom to send different messages, at different rates, to the encoder and the decoder. The converse proofs use a technique for verifying Markov relations via undirected graphs.

Index Terms—Rate-distortion, two-way rate distortion, undirected graphs, verification of Markov relations, Wyner–Ziv source coding.

I. INTRODUCTION

IN this paper, we consider the problem of two-way source encoding with a fidelity criterion in a situation where both users receive a common message from a helper. The problem is presented in Fig. 1. Note that the case in which the helper is absent was introduced and solved by Kaspi [1].

The encoding and decoding is done in blocks of length n. The communication protocol is that Helper Y first sends a common message at rate R_1 to User X and to User Z, and then User Z sends a message at rate R_2 to User X, and finally, User X sends a message to User Z at rate R_3. Note that user Z sends his message after it received only one message, while Sender X sends its message after it received two messages. We assume that the sources and the helper sequences are i.i.d. and form the Markov chain $Y - X - Z$. User X receives two messages (one from the helper and one from User Z) and reconstructs the source Z^n. We assume that the fidelity (or distortion) is of the form $\mathbb{E}[\frac{1}{n}\sum_{i=1}^n d_z(Z_i, \hat{Z}_i)]$ and that this term should be less than a threshold D_z. User Z also receives two messages (one from the
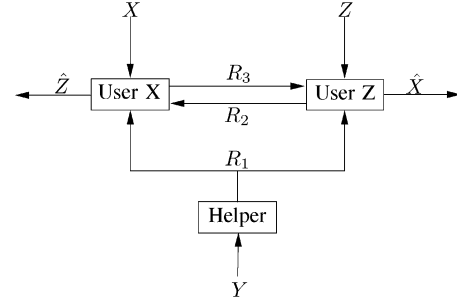
Fig. 1. Two-way rate distortion problem with a helper. First Helper Y sends a common message to User X and to User Z, then User Z sends a message to User X, and finally User X sends a message to User Z. The goal is that User X will reconstruct the sequence Z^n within a fidelity criterion $\mathbb{E}[\frac{1}{n}\sum_{i=1}^n d_z(Z_i, \hat{Z}_i)] \leq D_z$, and User Z will reconstruct the source X^n within a fidelity criterion $\frac{1}{n}\mathbb{E}[\sum_{i=1}^n d_x(X_i, \hat{X}_i)] \leq D_x$. We assume that the side information Y and the two sources X, Z are i.i.d. and form the Markov chain $Y - X - Z$.

helper and one from User X) and reconstructs the source X^n. The reconstruction $\hat{X}^n$ must lie within a fidelity criterion of the form $\frac{1}{n}\mathbb{E}[\sum_{i=1}^n d_x(X_i, \hat{X}_i)] \leq D_x$.

A practical scenario where this setting may occur is in a peer-to-peer network, where node Y (the helper) has a noisy observation of another node X. Node X and Z need to exchange information and the helper can multicast to both nodes. In particular, we discuss and obtain a closed-form characterization for the quadratic Gaussian case, which is frequently used as a simple model, where limited loss of information is allowed [2].

Our main result in this paper is that the achievable region for this problem is given by $\mathcal{R}(D_x, D_z)$, which is defined as the set of all rate triples (R_1, R_2, R_3) that satisfy

$$R_1 \geq I(Y; U \,|\, Z) \tag{1}$$
$$R_2 \geq I(Z; V \,|\, U, X) \tag{2}$$
$$R_3 \geq I(X; W \,|\, U, V, Z) \tag{3}$$

for some joint distribution of the form

$$
\begin{aligned}
&p(x, y, z, u, v, w) \\
&\quad = p(x, y)p(z \,|\, x)p(u \,|\, y)p(v \,|\, u, z)p(w \,|\, u, v, x) \quad (4)
\end{aligned}
$$

where U, V, and W are auxiliary random variables with bounded cardinality. The reconstruction variable $\hat{Z}$ is a deterministic function of the triple (U, V, X), and the reconstruction $\hat{X}$ is a deterministic function of the triple (U, W, Z) such that

$$
\begin{aligned}
\mathbb{E} d_x(X, \hat{X}(U, W, Z)) &\leq D_x \\
\mathbb{E} d_z(Z, \hat{Z}(U, V, X)) &\leq D_z. \quad (5)
\end{aligned}
$$

The main insight gained from this region is that the helper may use a code based on binning that is designed for a decoder

with side information, as in [3]. User X and User Z do not have the same side information, but it is sufficient to design the helper's code assuming that the side information at the decoder is the one that is "further" in the Markov chain, namely, Z. Since a distribution of the form (4) implies that $I(U; Z) \leq I(U; X)$, a Wyner–Ziv code at rate $R_1 \geq I(Y; U \mid Z)$ would be decoded successfully both by User Z and by User X. Once the helper's message has been decoded by both users, a two-way source coding is performed where both users have additional side information U^n.

Several papers on related problems have appeared in the past in the literature. Wyner [4] studied a problem of network source coding with compressed side information that is provided only to the decoders. A special case of his model is the system in Fig. 1 but without the memoryless side information Z and where the stream carrying the helper's message arrives only at the decoder (User Z). A full characterization of the achievable region can be concluded from the results of [4] for the special case where the source X has to be reconstructed losslessly. This problem was solved independently by Ahlswede and Körner in [5], but the extension of these results to the case of lossy reconstruction of X remains open. Kaspi [6] and Kaspi and Berger [7] derived an achievable region for a problem that contains the helper problem with degenerate Z as a special case. However, the converse part does not match. In [8], Vasudevan and Perron described a general rate distortion problem with encoder breakdown and there they solved the case where in Fig. 1 one of the sources is a constant.[1]

Reference [10] solved the multiterminal source coding problem where one of the two sources needs to be reconstructed perfectly and the other source needs to be reconstructed with a fidelity criterion. Oohama solved the multiterminal source coding case for the two [11] and $L + 1$ [12] Gaussian sources, in which only one source needs to be reconstructed with a mean square error, that is, the other L sources are helpers. More recently, Wagner, Tavildar, and Viswanath characterized the region where both sources [13] or $L + 1$ sources [14] need to be reconstructed at the decoder with a mean square error criterion.

In [1], Kaspi has introduced a multistage communication between two users, where each user may transmit up to K messages to the other user that depends on the source and previous received messages. In this paper, we also consider the multistage source coding with a common helper. The case where a helper is absent and the communication between the users is via memoryless channels was recently solved by Maor and Merhav [15] where they showed that a source channel separation theorem holds.

The remainder of the paper is organized as follows. In Section II, we present a technique for verifying Markov relations between random variables based on undirected graphs. The technique is used throughout the converse proofs. The problem definition and the achievable region for two way rate distortion problem with a common helper are presented in Section III. Then we consider two special cases, first in Section IV, we consider the case of $R_2 = 0$ and $D_z = \infty$,

and in Section V, we consider $R_3 = 0$ and $D_x = \infty$. The proofs of these two special cases provide the insight and the tricks that are used in the proof of the general two-way rate distortion problem with a helper. The proof of the achievable region for the two-way rate distortion problem with a helper is given in Section VI and it is extended to a multistage two way rate distortion with a helper in Section VII. In Section VIII, we consider the Gauissan instance of the problem and derive the region explicitly. In Section IX, we return to the special case where $R_2 = 0$ and $D_z = \infty$ and analyze the tradeoff between the bits from the helper and bits from source and gain insight for the case where the helper sends different messages to each user, which is an open problem.

II. PRELIMINARY: A TECHNIQUE FOR CHECKING MARKOV RELATIONS

Here we present a technique, based on undirected graphs, that provides a sufficient condition for establishing a Markov chain from a joint distribution. It's similar in spirit to techniques in [16, Ch 1.2], [17], [18, pp. 231] etc., but we present here a version that is most suitable for our particular converse applications. We derive it from first principles for its simplicity and for completeness.

Assume we have a set of random variables $(X_1, X_2, \ldots, X_N)$, where N is the size of the set. Without loss of generality, we assume that the joint distribution has the form

$$p(x^N) = f(x_{\mathcal{S}_1}) f(x_{\mathcal{S}_2}) \cdots f(x_{\mathcal{S}_K}) \qquad (6)$$

where $X_{\mathcal{S}_i} = \{X_j\}_{j \in \mathcal{S}_i}$, where $\mathcal{S}_i$ is a subset of $\{1, 2, \ldots, N\}$. The following graphical technique provides a sufficient condition for the Markov relation $X_{\mathcal{G}_1} - X_{\mathcal{G}_2} - X_{\mathcal{G}_3}$, where $X_{\mathcal{G}_i}$, $i = 1, 2, 3$ denote three disjoint subsets of X^N.

The technique comprises two steps:
1) draw an undirected graph where all the random variables X^N are nodes in the graph and for all $i = 1, 2, \ldots K$ draw edges between all the nodes $X_{\mathcal{S}_i}$;
2) if all paths in the graph from a node in $X_{\mathcal{G}_1}$ to a node in $X_{\mathcal{G}_3}$ pass through a node in $X_{\mathcal{G}_2}$, then the Markov chain $X_{\mathcal{G}_1} - X_{\mathcal{G}_2} - X_{\mathcal{G}_3}$ holds.

Example 1: Consider the joint distribution

$$p(x^2, y^2, z^2) = p(x_1, y_2) p(y_1, x_2) p(z_1 \mid x_1, x_2) p(z_2 \mid y_1). \qquad (7)$$

Fig. 2 illustrates the above technique for verifying the Markov relation $X_1 - X_2 - Z_2$. We conclude that since all the paths from X_1 to Z_2 pass through X_2, the Markov chain $X_1 - X_2 - Z_2$ holds.

The proof of the technique is based on the observation that if three random variables X, Y, Z have a joint distribution of the form $p(x, y, z) = f(x, y) f(y, z)$, then the Markov chain $X - Y - Z$ holds. The proof appears in Appendix A.

In the two way source coding without a helper [1], Kaspi has used the result [1, Lemma 2] which states that given a joint distribution of the form $p(x^2, y^2, z^2) = p(x_1, y_2) p(y_1, x_2) p(z_1 \mid x_1, x_2) p(z_2 \mid y_1, y_2, z_1)$, the Markov

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[1]The case where one of the sources is constant was also considered independently in [9].
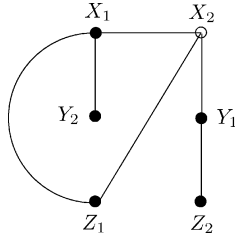
Fig. 2. The undirected graph that corresponds to the joint distribution given in (7). The Markov form $X_1 - X_2 - Z_2$ holds since all paths from X_1 to Z_2 pass through X_2. The node with the open circle, i.e., $\circ$, is the middle term in the Markov chain and all the other nodes are with solid circles, i.e., $\bullet$.

relation $X_2 - (Z_1, Z_2, X_1, Y_2) - Y_1$ holds.[2] In the presence of a helper, additional complicated Markov relations are needed to be proved, and, therefore, we developed the graphical technique.

III. PROBLEM DEFINITIONS AND MAIN RESULTS

Here we formally define the two-way rate-distortion problem with a helper and present a single-letter characterization of the achievable region. We use the regular definitions of rate distortion and we follow the notation of [19]. The source sequences $\{X_i \in \mathcal{X}, i = 1, 2, \ldots\}$, $\{Z_i \in \mathcal{Z}, i = 1, 2, \ldots\}$ and the side information sequence $\{Y_i \in \mathcal{Y}, i = 1, 2, \ldots\}$ are discrete random variables drawn from finite alphabets $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$, respectively. The random variables (X_i, Y_i, Z_i) are i.i.d. $\sim p(x, y, z)$. Let $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$ be the reconstruction alphabets, and $d_x : \mathcal{X} \times \hat{\mathcal{X}} \to [0, \infty)$, $d_z : \mathcal{Z} \times \hat{\mathcal{Z}} \to [0, \infty)$ be single letter distortion measures. Distortion between sequences is defined in the usual way

$$d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i)$$

$$d(z^n, \hat{z}^n) = \frac{1}{n} \sum_{i=1}^{n} d(z_i, \hat{z}_i). \tag{8}$$

Let $\mathcal{M}_i$, denote a set of positive integers $\{1, 2, \ldots, M_i\}$ for $i = 1, 2, 3$.

Definition 1: An $(n, M_1, M_2, M_3, D_x, D_z)$ code for two sources X and Z with helper Y consists of three encoders

$$\begin{aligned} f_1 &: \mathcal{Y}^n \to \mathcal{M}_1 \\ f_2 &: \mathcal{Z}^n \times \mathcal{M}_1 \to \mathcal{M}_2 \\ f_3 &: \mathcal{X}^n \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_3 \end{aligned} \tag{9}$$

and two decoders

$$\begin{aligned} g_2 &: \mathcal{X}^n \times \mathcal{M}_1 \times \mathcal{M}_2 \to \hat{\mathcal{Z}}^n \\ g_3 &: \mathcal{Z}^n \times \mathcal{M}_1 \times \mathcal{M}_3 \to \hat{\mathcal{X}}^n \end{aligned} \tag{10}$$

[2]The proof of this Markov relation in [1] is based on the chain rule of mutual information. One can also verify it using the undirected graph technique, presented here

such that

$$\mathbb{E}\left[\sum_{i=1}^{n} d_x(X_i, \hat{X}_i)\right] \leq D_x$$

$$\mathbb{E}\left[\sum_{i=1}^{n} d_z(Z_i, \hat{Z}_i)\right] \leq D_z \tag{11}$$

The rate triple (R_1, R_2, R_3) of the $(n, M_1, M_2, M_3, D_x, D_z)$ code is defined by

$$R_i = \frac{1}{n} \log M_i; \quad i = 1, 2, 3. \tag{12}$$

Definition 2: Given a distortion pair (D_x, D_z), a rate triple (R_1, R_2, R_3) is said to be *achievable* if, for any $\epsilon > 0$, and sufficiently large n, there exists an $(n, 2^{nR_1}, 2^{nR_2}, 2^{nR_3}, D_x + \epsilon, D_z + \epsilon)$ code for the sources X, Z with side information Y.

Definition 3: The (operational) achievable region $\mathcal{R}^O(D_x, D_z)$ of rate distortion with a helper known at the encoder and decoder is the closure of the set of all achievable rate pairs.

The next theorem is the main result of this work.

Theorem 1: In the tow-way rate distortion problem with a helper, as depicted in Fig. 1, where $Y - X - Z$

$$\mathcal{R}^O(D_x, D_z) = \mathcal{R}(D_x, D_z) \tag{13}$$

where the region $\mathcal{R}(D_x, D_z)$ is specified in (1)–(5).

Furthermore, the region $\mathcal{R}(D_x, D_z)$ satisfies the following properties, which are proved in Appendix B.

Lemma 2:
1) The region $\mathcal{R}(D_x, D_z)$ is convex.
2) To exhaust $\mathcal{R}(D_x, D_z)$, it is enough to restrict the alphabet of U, V, and W to satisfy

$$\begin{aligned} |\mathcal{U}| &\leq |\mathcal{Y}| + 4 \\ |\mathcal{V}| &\leq |\mathcal{Z}||\mathcal{U}| + 3 \\ |\mathcal{W}| &\leq |\mathcal{U}||\mathcal{V}||\mathcal{X}| + 1. \end{aligned} \tag{14}$$

Before proving the main result (Theorem 1), we would like to consider two special cases, first where $R_2 = 0$ and $D_z = \infty$ and second where $R_3 = 0$ and $D_x = \infty$. The main techniques and insight are gained through those special cases. Both cases are depicted in Fig. 3 where in the first case we assume the Markov form $Y - X - Z$ and in the second case we assume a Markov form $Y - Z - X$.

The proofs of these two cases are quite different. In the achievability of the first case, we use a Wyner–Ziv code that is designed only for the *decoder*, and in the achievability of the second case we use a Wyner–Ziv code that is designed only for the *encoder*. In the converse for the first case, the main idea is to observe that the achievable region does not increase by letting the encoder know Y, and in the converse of the second case the
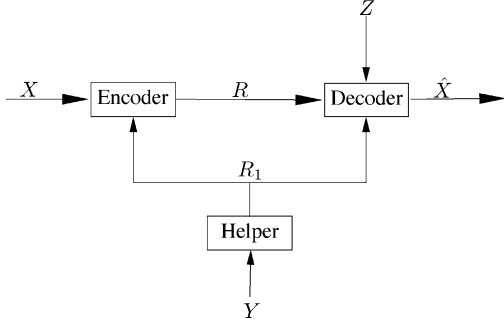
Fig. 3. Wyner–Ziv problem with a helper. We consider two cases; first the source X, Helper Y and the side information Z form the Markov chain $Y - X - Z$ and in the second case they form the Markov chain $Y - Z - X$.

main idea is to use the chain rule in two opposite directions, conditioning once on the past and once on the future.

IV. WYNER–ZIV WITH A HELPER WHEN Y-X-Z

In this section, we consider the rate distortion problem with a helper and additional side information Z, known only to the decoder, as shown in Fig. 3. We also assume that the source X, the helper Y, and the side information Z, form the Markov chain $Y - X - Z$. This setting corresponds to the case where $R_2 = 0$ and $D_z = \infty$. Let us denote by $\mathcal{R}^O_{Y-X-Z}(D)$ the (operational) achievable region $\mathcal{R}^O(D_x = D, D_z = \infty)$.

We now present our main result of this section. Let $\mathcal{R}_{Y-X-Z}(D)$ be the set of all rate pairs (R, R_1) that satisfy

$$R_1 \geq I(U; Y \mid Z) \tag{15}$$
$$R \geq I(X; W \mid U, Z) \tag{16}$$

for some joint distribution of the form

$$p(x, y, z, u, v) = p(x, y)p(z \mid x)p(u \mid y)p(w \mid x, u) \tag{17}$$
$$\mathbb{E}d_x(X, \hat{X}(U, W, Z)) \leq D \tag{18}$$

where W and V are auxiliary random variables, and the reconstruction variable $\hat{X}$ is a deterministic function of the triple (U, W, Z). The next lemma states properties of $\mathcal{R}_{Y-X-Z}(D)$. It is the analog of Lemma 2 and the proof is omitted.

Lemma 3:
1) The region $\mathcal{R}_{Y-X-Z}(D)$ is convex.
2) To exhaust $\mathcal{R}_{Y-X-Z}(D)$, it is enough to restrict the alphabets of V and U to satisfy

$$|\mathcal{U}| \leq |\mathcal{Y}| + 2$$
$$|\mathcal{W}| \leq |\mathcal{X}|(|\mathcal{Y}| + 2) + 1. \tag{19}$$

Theorem 4: The achievable rate region for the setting illustrated in Fig. 3, where X, Y, Z are i.i.d. random variables forming the Markov chain $Y - X - Z$ is

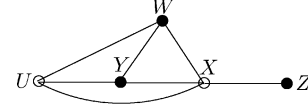$$\mathcal{R}^O_{Y-X-Z}(D) = \mathcal{R}_{Y-X-Z}(D). \tag{20}$$

Fig. 4. Graphical proof of the Markov chain $W - (X, U) - Z$. The undirected graph corresponds to the joint distribution given in (22), i.e., $\bar{p}(x, y, z, u, v, w) = p(x, y)p(z \mid x)p(u \mid y)p(w \mid u, x, y)$. The Markov chain holds since there is no path from Z to W that does not pass through (X, U).

Let us define an additional region $\bar{\mathcal{R}}_{Y-X-Z}(D)$ the same as $\mathcal{R}_{Y-X-Z}(D)$ but the term $p(w \mid x, u)$ in (17) is replaced by $p(w \mid x, u, y)$, i.e.

$$p(x, y, z, u, w) = p(x, y)p(z \mid x)p(u \mid y)p(w \mid x, u, y). \tag{21}$$

In the proof of Theorem 4, we show that $\mathcal{R}_{Y-X-Z}(D)$ is achievable and that $\bar{\mathcal{R}}_{Y-X-Z}(D)$ is an outer bound, and we conclude the proof by applying the following lemma, which states that the two regions are equal.

Lemma 5: $\bar{\mathcal{R}}_{Y-X-Z}(D) = \mathcal{R}_{Y-X-Z}(D)$.

Proof: Trivially, we have $\bar{\mathcal{R}}_{Y-X-Z}(D) \supseteq \mathcal{R}(D \mid Z)$. Now we prove that $\bar{\mathcal{R}}_{Y-X-Z}(D) \subseteq \mathcal{R}_{Y-X-Z}(D)$. Let $(R, R_1) \in \bar{\mathcal{R}}_{Y-X-Z}(D)$, and

$$\bar{p}(x, y, z, u, w) = p(x, y)p(z \mid x)p(u \mid y)\bar{p}(w \mid x, u, y) \tag{22}$$

be a distribution that satisfies (15), (16), and (18). Now we show that there exists a distribution of the form (17) such that (16), (15), and (18) hold.

Let

$$p(x, y, z, u, w) = p(x, y, z)p(u \mid y)\bar{p}(w \mid x, u) \tag{23}$$

where $\bar{p}(w \mid x, u)$ is induced by $\bar{p}(x, y, z, u, w)$. We now show that the terms $I(U; Y \mid Z)$, $I(X; W \mid Z, U)$ and $\mathbb{E}d(X, \hat{X}(U, W, Z))$ are the same whether we evaluate them by the joint distribution $p(x, y, z, u, w)$ of (23), or by $\bar{p}(x, y, z, u, w)$; hence $(R, R_1) \in \mathcal{R}_{Y-X-Z}(D)$. In order to show that the terms above are the same it is enough to show that the marginal distributions $p(y, z, u)$ and $p(x, z, u, w)$ induced by $p(x, y, z, u, w)$ are equal to the marginal distributions $\bar{p}(y, z, u)$ and $\bar{p}(x, z, u, w)$ induced by $\bar{p}(x, y, z, u, w)$. Clearly $p(y, u, z) = \bar{p}(y, u, z)$. In the rest of the proof we show $p(x, z, u, w) = \bar{p}(x, z, u, w)$.

A distribution of the form $\bar{p}(x, y, z, u, w)$ as given in (22) implies that the Markov chain $W - (X, U) - Z$ holds as shown in Fig. 4. Therefore $\bar{p}(w \mid x, u, z) = \bar{p}(w \mid x, u)$. Now consider $\bar{p}(x, z, u, w) = \bar{p}(x, z, u)\bar{p}(w \mid x, u)$, and since $\bar{p}(x, z, u) = p(x, z, u)$ and $\bar{p}(w \mid x, u) = p(w \mid x, u)$ we conclude that $\bar{p}(x, z, u, w) = p(x, z, u, w)$. ∎

Proof of Theorem 4: **Achievability:** The proof follows classical arguments, and therefore the technical details will be omitted. We describe only the coding structure and the associated Markov conditions. Note that the condition (17)

in the definition of $\mathcal{R}_{Y-X-Z}(D)$, implies the Markov chain $U-Y-X-Z$. The helper (encoder of Y) employs Wyner–Ziv coding with decoder side information Z and external random variable U, as seen from (15). The Markov conditions required for such coding, $U-Y-Z$, are satisfied, hence the source decoder, can recover the codewords constructed from U. Moreover, since (17) implies $U-Y-X-Z$, the encoder of X can also reconstruct U (this is the point where the Markov assumption $Y-X-Z$ is used). Therefore in the coding/decoding scheme of X, U serves as side information available at both sides. The source (X) encoder now employs Wyner–Ziv coding for X, with decoder side information Z, coding random variable W, and U available at both sides. The Markov conditions needed for this scheme are $W-(X,U)-Z$, which again are satisfied by (17). The rate needed for this coding is $I(X;W\,|\,U,Z)$, reflected in the bound on R in (16). Once the two codes (helper and source code) are decoded, the destination can use all the available random variables, U, W, and the side information Z, to construct $\hat{X}$.

Converse: Assume that we have an $(n, M_1 = 2^{nR_1}, M_2 = 1, M_3 = 2^{nR}, D_x = D, D_z = \infty)$ code as in Definition 4. We will show the existence of a triple $(U, W, \hat{X})$ that satisfy (15)–(18). Denote $T_1 = f_1(Y^n) \in \{1, \ldots, 2^{nR_1}\}$, and $T = f_3(X^n, T_1) \in \{1, \ldots, 2^{nR}\}$. Then

$$
\begin{aligned}
nR_1 &\geq H(T_1) \\
&\geq H(T_1 \,|\, Z^n) \\
&\geq I(Y^n; T_1 \,|\, Z^n) \\
&= \sum_{i=1}^{n} H(Y_i \,|\, Z_i) - H(Y_i \,|\, Y^{i-1}, T_1, Z^n) \\
&\overset{(a)}{=} \sum_{i=1}^{n} H(Y_i \,|\, Z_i) - H(Y_i \,|\, X^{i-1}, Y^{i-1}, T_1, Z^n) \\
&\geq \sum_{i=1}^{n} H(Y_i \,|\, Z_i) - H(Y_i \,|\, X^{i-1}, T_1, Z^n) \qquad (24)
\end{aligned}
$$

where equality (a) is due to the Markov form $Y_i - (Y^{i-1}, f_1(Y^n), Z^n) - X^{i-1}$. Furthermore

$$
\begin{aligned}
nR &\geq H(T) \\
&\geq H(T \,|\, T_1, Z^n) \\
&\geq I(X^n; T \,|\, T_1, Z^n) \\
&= \sum_{i=1}^{n} H(X_i \,|\, T_1, Z^n, X^{i-1}) \\
&\quad - H(X_i \,|\, T, T_1, Z^n, X^{i-1}). \qquad (25)
\end{aligned}
$$

Now, let $W_i \triangleq T$ and $U_i \triangleq (X^{i-1}, Z^{n\backslash i}, T_1)$, where $Z^{n\backslash i}$ denotes the vector Z^n without the ith element, i.e., (Z^{i-1}, Z_{i+1}^n). Then (24) and (25) become

$$
R_1 \geq \frac{1}{n} \sum_{i=1}^{n} I(Y_i; U_i \,|\, Z_i)
$$

$$
R \geq \frac{1}{n} \sum_{i=1}^{n} I(X_i; W_i \,|\, U_i, Z_i). \qquad (26)
$$

Now we observe that the Markov chain $U_i - Y_i - (X_i, Z_i)$ holds since we have $(X^{i-1}, Z^{n\backslash i}, T_1(Y^n)) - Y_i - (X_i, Z_i)$. Also the Markov chain $W_i - (U_i, X_i, Y_i) - Z_i$ holds since $T(T_1, X^n) - (X^i, Y_i, T_1(Y^n), Z^{n\backslash i}) - Z_i$. The reconstruction at time i, i.e., $\hat{X}_i$, is a deterministic function of (Z^n, T, T_1), and, in particular, it is a deterministic function of (U_i, W_i, Z_i). Finally, let Q be a random variable independent of X^n, Y^n, Z^n, and uniformly distributed over the set $\{1, 2, 3, \ldots, n\}$. Define the random variables $U \triangleq (Q, U_Q)$, $W \triangleq (Q, W_Q)$, and $\hat{X} \triangleq (\hat{X}_Q)$ ($\hat{X}_Q$ is a short notation for time sharing over the estimators). The Markov relations $U - Y - (X, Z)$ and $W - (X, U, Y) - Z$, the inequality $\mathbb{E}d(X, \hat{X}) = \sum_{i=1}^{n} \frac{1}{n} \mathbb{E}d(X, \hat{X}_i) \leq D$, the fact that $\hat{X}$ is a deterministic function of (U, W, Z), and the inequalities $R_1 \geq I(Y; U \,|\, Z)$ and $R \geq I(X, Y; W \,|\, U, Z)$ (implied by (26)), imply that $(R, R_1) \in \bar{\mathcal{R}}_{Y-X-Z}(D)$, completing the proof by Lemma 5. ∎

V. WYNER–ZIV WITH A HELPER WHERE $Y - Z - X$

Consider the rate-distortion problem with side information and helper as illustrated in Fig. 3, where the random variables X, Y, Z form the Markov chain $Y - Z - X$. This setting corresponds to the case where $R_3 = 0$ and exchanging between X and Z. Let us denote by $\mathcal{R}_{Y-Z-X}^O(D)$ the (operational) achievable region.

Let $\mathcal{R}_{Y-Z-X}(D)$ be the set of all rate pairs (R, R_1) that satisfy

$$
R_1 \geq I(U; Y \,|\, X) \qquad (27)
$$

$$
R \geq I(X; V \,|\, U, Z) \qquad (28)
$$

for some joint distribution of the form

$$
p(x, y, z, u, v) = p(z, y)p(x \,|\, z)p(u \,|\, y)p(v \,|\, x, u) \qquad (29)
$$

$$
\mathbb{E}d(X, \hat{X}(U, V, Z)) \leq D \qquad (30)
$$

where U and V are auxiliary random variables, and the reconstruction variable $\hat{X}$ is a deterministic function of the triple (U, V, Z). The next lemma states properties of $\mathcal{R}_{Y-Z-X}(D)$. It is the analog of Lemma 2 and, therefore, omitted.

Lemma 6:
1) The region $\mathcal{R}_{Y-Z-X}(D)$ is convex
2) To exhaust $\mathcal{R}_{Y-Z-X}(D)$, it is enough to restrict the alphabets of V and U to satisfy

$$
|\mathcal{U}| \leq |\mathcal{Y}| + 2
$$

$$
|\mathcal{V}| \leq |\mathcal{X}|(|\mathcal{Y}| + 2) + 1. \qquad (31)
$$

Theorem 7: The achievable rate region for the setting illustrated in Fig. 3, where X_i, Y_i, Z_i are i.i.d. triplets distributed according to the random variables X, Y, Z forming the Markov chain $Y - Z - X$ is

$$
\mathcal{R}_{Y-Z-X}^O(D) = \mathcal{R}_{Y-Z-X}(D). \qquad (32)
$$

Proof: **Achievability:** The proof follows classical arguments, and therefore the technical details will be omitted. We describe only the coding structure and the associated Markov conditions. The helper (encoder of Y) employs Wyner–Ziv coding

with decoder side information X and external random variable U, as seen from (27). The Markov conditions required for such coding, $U - Y - X$, are satisfied, hence the source encoder, can recover the codewords constructed from U. Moreover, since (29) implies $U - Y - Z - X$, the decoder, at the destination, can also reconstruct U. Therefore, in the coding/decoding scheme of X, U serves as side information available at both sides. The source X encoder now employs Wyner–Ziv coding for X, with decoder side information Z, coding random variable V, and U available at both sides. The Markov conditions needed for this scheme are $V - (X, U) - Z$, which again are satisfied by (29). The rate needed for this coding is $I(X; V \mid U, Z)$, reflected in the bound on R in (28). Once the two codes (helper and source code) are decoded, the destination can use all the available random variables, U, V, and the side information Z, to construct $\hat{X}$.

Converse: Assume that we have a code for a source X with helper Y and side information Z at rate (R_1, R). We will show the existence of a triple $(U, V, \hat{X})$ that satisfy (27)–(30). Denote $T_1(Y^n) \in \{1, \ldots, 2^{nR_1}\}$, and $T(X^n, T1) \in \{1, \ldots, 2^{nR}\}$. Then

$$
\begin{aligned}
nR_1 &\geq H(T_1) \\
&\geq H(T_1 \mid X^n) \\
&= I(Y^n; T_1 \mid X^n) \\
&= \sum_{i=1}^n H(Y_i \mid X_i) - H(Y_i \mid Y^{i-1}, T_1, X^n) \\
&\overset{(a)}{=} \sum_{i=1}^n H(Y_i \mid X_i) - H\left(Y_i \mid Y^{i-1}, T_1, X_{i+1}^n, X_i\right) \\
&\overset{(b)}{=} \sum_{i=1}^n H(Y_i \mid X_i) \\
&\quad - H\left(Y_i \mid Y^{i-1}, T_1, X_{i+1}^n, X_i, Z^{i-1}\right) \\
&\overset{(c)}{\geq} \sum_{i=1}^n H(Y_i \mid X_i) \\
&\quad - H\left(Y_i \mid T_1, X_{i+1}^n, X_i, Z^{i-1}\right)
\end{aligned}
\tag{33}
$$

where (a) and (b) follow from the Markov chain $Y_i - (Y^{i-1}, T_1(Y^n), X_i^n) - (X^{i-1}, Z^{i-1})$ (see Fig. 5 for the proof), and (c) follows from the fact that conditioning reduces entropy. Consider

$$
\begin{aligned}
nR &\geq H(T) \\
&\geq H(T \mid T_1, Z^n) \\
&= I(X^n; T \mid T_1, Z^n) \\
&= \sum_{i=1}^n H\left(X_i \mid X_{i+1}^n, T_1, Z^n\right) \\
&\quad - H\left(X_i \mid X_{i+1}^n, T_1, Z^n, T\right) \\
&\overset{(a)}{=} \sum_{i=1}^n H\left(X_i \mid X_{i+1}^n, T_1, Z^{i-1}, Z_i\right) \\
&\quad - H\left(X_i \mid X_{i+1}^n, T_1, Z^n, T\right) \\
&\overset{(b)}{\geq} \sum_{i=1}^n H\left(X_i \mid X_{i+1}^n, T_1, Z^{i-1}, Z_i\right) \\
&\quad - H\left(X_i \mid X_{i+1}^n, T_1, Z^{i-1}, Z_i, T\right)
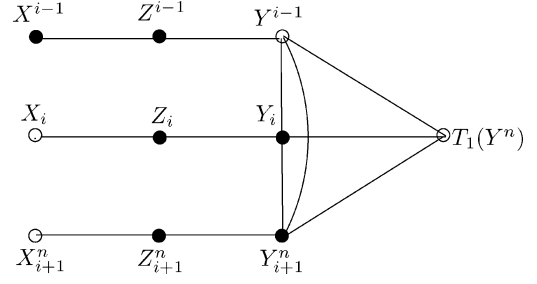\end{aligned}
\tag{34}
$$

Fig. 5. Graphical proof of the Markov chain $Y_i - (Y^{i-1}, T_1(Y^n), X_i^n) - (X^{i-1}, Z^{i-1})$. The undirected graph corresponds to the joint distribution $p(x^{i-1}, z^{i-1}) p(y^{i-1} \mid z^{i-1}) p(x_i, z_i) p(y_i \mid z_i) p(x_{i+1}^n, z_{i+1}^n) p(y_{i+1}^n \mid z_{i+1}^n) p(t_1 \mid y^n)$. The Markov chain holds since all paths from Y_i to X^{i-1}, Z^{i-1} pass through $(Y^{i-1}, T_1(Y^n), X_i^n)$. The nodes with the open circle, i.e., $\circ$, constitute the middle term in the Markov chain, i.e., $(Y^{i-1}, T_1(Y^n), X_i^n)$ and all the other nodes are with solid circles, i.e., $\bullet$. The nodes Y^{i-1}, Y_i, Y_{i+1}^n and T_1 are connected due to the term $p(t_1 \mid y^n)$.
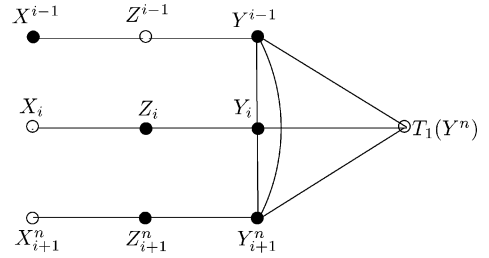

Fig. 6. Graphical proof of the Markov chain $X^{i-1} - (Z^{i-1}, T_1(Y^n), X_i^n) - (Z_i, Y_i)$, which implies $V_i - (X_i, U_i) - (Z_i, Y_i)$. The undirected graph corresponds to the joint distribution $p(x^{i-1}, z^{i-1}) p(y^{i-1} \mid z^{i-1}) p(x_i, z_i) p(y_i \mid z_i) p(x_{i+1}^n, z_{i+1}^n) p(y_{i+1}^n \mid z_{i+1}^n) p(t_1 \mid y^n)$. The Markov chain holds since all paths from X^{i-1} to (Z_i, Y_i) pass through $(Z^{i-1}, T_1(Y^n), X_i^n)$.

where (a) is due to the Markov chain $X_i - (X_{i+1}^n, T_1(Y^n), Z^i) - Z_{i+1}^n$ (this can be seen from Fig. 5 since all paths from X_i to Z_{i+1}^n goes through Z_i), and (b) is due to the fact that conditioning reduces entropy. Now let us denote $U_i \triangleq Z^{i-1}, T_1(Y^n), X_{i+1}^n$, and $V_i \triangleq T(X^n, T_1)$. The Markov chains $U_i - Y_i - (X_i, Z_i)$ and $V_i - (X_i, U_i) - (Z_i, Y_i)$ hold (see Fig. 6 for the proof of the last Markov relation).

Next, we need to show that there exists a sequence of function $\hat{X}_i(U_i, V_i, Z_i)$ such that

$$
\frac{1}{n} \sum_{i=1}^n \mathbb{E}[d(X_i, \hat{X}_i(U_i, V_i, Z_i))] \leq D.
\tag{35}
$$

By assumption we know that there exists a sequence of functions $\hat{X}_i(T, T_1, Z^n)$ such that $\sum_{i=1}^n \mathbb{E}[d(X_i, \hat{X}_i(T, T_1, Z^n))] \leq nD$, and trivially this implies that there exists a sequence of functions $\hat{X}_i(X_{i+1}^n, T, T_1, Z^n)$ such that

$$
\sum_{i=1}^n \mathbb{E}\left[d\left(X_i, \hat{X}_i\left(X_{i+1}^n, T, T_1, Z^i, Z_{i+1}^n\right)\right)\right] \leq D.
\tag{36}
$$

Note that the Markov chain $X_i - (X_{i+1}^n, T_1, Z^i, T) - Z_{i+1}^n$ holds (see Fig. 7 for the proof), hence there exists a function deterministic function f' and a random variable U',
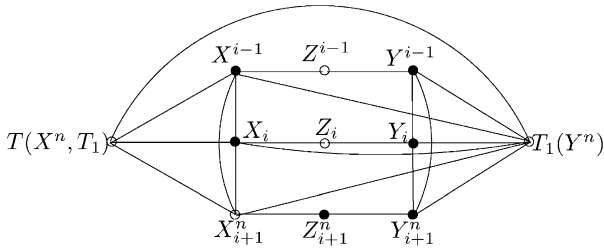
Fig. 7. Graphical proof of the Markov chain $X_i - (X_{i+1}^n, T_1, Z^i, T) - Z_{i+1}^n$. The undirected graph corresponds to the joint distribution $p(x^{i-1}, z^{i-1})p(y^{i-1} \mid z^{i-1})p(x_i, z_i)p(y_i \mid z_i)p(x_{i+1}^n, z_{i+1}^n)p(y_{i+1}^n \mid z_{i+1}^n)$ $p(t_1 \mid y^n)p(t \mid x^n, t_1)$. The Markov chain holds since all paths from X_i to Z_{i+1}^n pass through (X_{i+1}^n, T_1, Z^i, T).

which is independent of $(X_i, X_{i+1}^n, T_1, Z^i, T)$ such that $Z_{i+1}^n = f'(U', X_{i+1}^n, T_1, Z^i, T)$. Therefore

$$\sum_{i=1}^n \mathbb{E}\left[d\left(X_i, \hat{X}_i\left(X_{i+1}^n, T, T_1, Z^i, Z_{i+1}^n\right)\right)\right]$$
$$= \sum_{u'} P(u') \sum_{i=1}^n \mathbb{E}\left[d\left(X_i, \hat{X}_i\left(X_{i+1}^n, T, T_1, Z^i, f'(u', X_{i+1}^n, T_1, Z^i, T)\right)\right)\right]$$
$$\geq \min_{u'} \sum_{i=1}^n \mathbb{E}\left[d\left(X_i, \hat{X}_i\left(X_{i+1}^n, T, T_1, Z^i, f'(u', X_{i+1}^n, T_1, Z^i, T)\right)\right)\right]$$
$$(37)$$

and since the RHS of (37) includes only the random variables (X_{i+1}^n, T, T_1, Z^i) we conclude that there exists a sequence of functions $\{\hat{X}_i(X_{i+1}^n, T, T_1, Z^i)\}$ for which (35) holds.

Finally, let Q be a random variable independent of X^n, Y^n, Z^n, and uniformly distributed over the set $\{1, 2, 3, \ldots, n\}$. Define the random variables $U \triangleq (Q, U_Q)$, $W \triangleq (Q, W_Q)$, and $\hat{X} \triangleq \hat{X}_Q$ ($\hat{X}_Q$ is a short notation for time sharing over the estimators). Then (33)–(35) implies that (27)–(30) hold. ∎

VI. PROOF OF THEOREM 1

In this section we prove Theorem 1, which states that the (operational) achievable region $\mathcal{R}^O(D_x, D_z)$ of the two-way source coding with helper problem as in Fig. 1 equals $\mathcal{R}(D_x, D_z)$. In the converse proof, we use the ideas used in proving the converses of Theorems 4 and 7. Namely, we will use the chain rule based on the past and future, and will show that $\mathcal{R}^O(D_x, D_z) \subseteq \bar{\mathcal{R}}(D_x, D_z)$, where $\bar{\mathcal{R}}(D_x, D_z)$ is defined as $\mathcal{R}(D_x, D_z)$ in (1)–(5) but with one difference: the term $p(w \mid u, v, x)$ in (4) should be replaced by $p(w \mid u, v, x, y)$, i.e.

$$p(x, y, z, u, v, w)$$
$$= p(x, y)p(z \mid x)p(u \mid y)p(v \mid u, z)p(w \mid u, v, x, y). \quad (38)$$

The following lemma states that the two regions $\bar{\mathcal{R}}(D_x, D_z)$ and $\mathcal{R}(D_x, D_z)$ are equal.

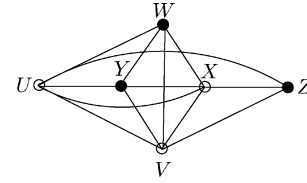*Lemma 8:* $\bar{\mathcal{R}}(D_x, D_z) = \mathcal{R}(D_x, D_z)$.

Fig. 8. Graphical proof of the Markov chain $W - (X, U, V) - Z$. The undirected graph corresponds to the joint distribution given in (39), i.e., $\bar{p}(x, y, z, u, v, w) = p(x, y)p(z \mid x)p(u \mid y)p(v \mid u, z)\bar{p}(w \mid u, v, x, y)$. The Markov chain holds since there is no path from Z to W that does not pass through (X, U, V).

Proof: Trivially, we have $\bar{\mathcal{R}}(D_x, D_z) \supseteq \mathcal{R}(D_z, D_z)$. Now we prove that $\bar{\mathcal{R}}(D_x, D_z) \subseteq \mathcal{R}(D_x, D_z)$. Let $(R_1, R_2, R_3) \in \bar{\mathcal{R}}(D_x, D_z)$, and

$$\bar{p}(x, y, z, u, v, w)$$
$$= p(x, y)p(z \mid x)p(u \mid y)p(v \mid u, z)\bar{p}(w \mid u, v, x, y) \quad (39)$$

be a distribution that satisfies (1)–(3) and (5). Next we show that there exists a distribution of the form of (4) (which is explicitly given in (39)) such that (1)–(3) and (5) hold. Let

$$p(x, y, z, u, v, w)$$
$$= p(x, y)p(z \mid x)p(u \mid y)p(v \mid u, z)\bar{p}(w \mid u, v, x) \quad (40)$$

where $\bar{p}(w \mid u, v, x)$ is induced by $\bar{p}(x, y, z, u, v)$. We show that all the terms in (1)–(3) and (5) i.e., $I(Y; U \mid Z)$, $I(Z; V \mid U, X)$, $\mathbb{E}d_z(Z, \hat{Z}(U, V, X))$, $I(X; W \mid U, V, Z)$, and $\mathbb{E}d_x(X, \hat{X}(U, W, Z))$ are the same whether we evaluate them by the joint distribution $p(x, y, z, u, v)$ of (40), or by $\bar{p}(x, y, z, u, v, w)$ of (39); hence $(R_1, R_2, R_3) \in \mathcal{R}(D_x, D_z)$. In order to show that the terms above are the same it is enough to show that the marginal distributions $p(x, y, z, u, v)$ and $p(x, z, u, v, w)$ induced by $p(x, y, z, u, v, w)$ are equal to the marginal distributions $\bar{p}(x, y, z, u, v)$ and $\bar{p}(x, z, u, v, w)$ induced by $\bar{p}(x, y, z, u, v, w)$. Clearly $p(x, y, z, u, v) = \bar{p}(x, y, z, u, v)$. In the rest of the proof we show $p(x, z, u, v, w) = \bar{p}(x, z, u, v, w)$.

A distribution of the form $\bar{p}(x, y, z, u, v, w)$ as given in (39) implies that the Markov chain $W - (X, U, V) - Z$ holds (see Fig. 8 for the proof). Therefore $\bar{p}(w \mid u, x, v, z) = \bar{p}(w \mid u, x, v)$. Since $\bar{p}(x, z, u, v, w) = \bar{p}(x, z, v, u)\bar{p}(w \mid x, u, v)$, and since $\bar{p}(x, z, v, u) = p(x, z, v, u)$ and $\bar{p}(w \mid x, u, v) = p(w \mid x, u, v)$ we conclude that $\bar{p}(x, z, u, v, w) = p(x, z, u, v, w)$. ∎

Proof of Theorem 1: **Achievability:** The achievability scheme is based on the fact that for the two special cases considered above, namely $R_2 = 0$ and $R_3 = 0$, the coding scheme for the helper was based on a Wyner–Ziv scheme, where the side information at the decoder is the random variable that is "further" in the Markov chain $Y - X - Z$, namely Z. The helper (encoder of Y) employs Wyner–Ziv coding with decoder side information Z and external random variable U, as seen from (1), i.e., $R_1 \geq I(Y; U \mid Z)$. The Markov conditions required for such coding, $U - Y - Z$, are satisfied, hence the source

decoder, at the destination, can recover the codewords constructed from U. Moreover, since (29) implies $U - Y - X - Z$, the encoder of X can also reconstruct U. Therefore, in the coding/decoding scheme of X, U serves as side information available at both sides. The source Z encoder now employs Wyner–Ziv coding for Z, with decoder side information X, coding random variable V, and U available at both sides. The Markov conditions needed for this scheme are $V - (Z, U) - X$, which again are satisfied by (4). The rate needed for this coding is $I(Z; V \mid U, X)$, reflected in the bound on R_2 in (2). Finally, the source X encoder now employs Wyner–Ziv coding for X, with decoder side information Z, coding random variable W, and U, V available at both sides. The Markov conditions needed for this scheme are $W - (X, U, V) - Z$, which again are satisfied by (4). The rate needed for this coding is $I(X; W \mid U, V, Z)$, reflected in the bound on R_3 in (3). Once the codes are decoded, the destination can use all the available random variables, (U, V, X) at User X, and, (U, W, Z) at User Z, to construct $\hat{Z}$ and $\hat{X}$, respectively.

Converse: Assume that we have a $(n, M_1, M_2, M_3, D_x, D_z)$ code. We now show the existence of a triple $(U, V, W, \hat{X}, \hat{Z})$ that satisfy (1)–(5). Denote $T_1 = f_1(Y^n)$, $T_2 = f_2(Z^n, T_1)$, and $T_3 = f_3(X^n, T_2, T_1)$. Then using the same arguments as in (33) and (34) (just exchanging between X and Z), we obtain

$$nR_1 \geq \sum_{i=1}^{n} H(Y_i \mid Z_i) - H\left(Y_i \mid X^{i-1}, T_1, Z_i^n\right) \quad (41)$$

$$nR_2 \geq \sum_{i=1}^{n} H\left(Z_i \mid Z_{i+1}^n, T_1, X^{i-1}, X_i\right)$$
$$- H\left(Z_i \mid Z_{i+1}^n, T_1, X^{i-1}, X_i, T_2\right) \quad (42)$$

respectively. For upper-bounding R_3, consider

$$
\begin{aligned}
nR_3 &\geq H(T_3) \\
&\geq H(T_3 \mid T_1, T_2, Z^n) \\
&\geq I(X^n; T_3 \mid T_1, T_2, Z^n) \\
&= \sum_{i=1}^{n} H(X_i \mid X^{i-1}, Z^n, T_1, T_2) \\
&\quad - H(X_i \mid X^{i-1}, Z^n, T_1, T_2, T_3) \\
&\overset{(a)}{=} \sum_{i=1}^{n} H\left(X_i \mid X^{i-1}, Z_i^n, T_1, T_2\right) \\
&\quad - H(X_i \mid X^{i-1}, Z^n, T_1, T_2, T_3) \\
&\geq \sum_{i=1}^{n} H\left(X_i \mid X^{i-1}, Z_i^n, T_1, T_2\right) \\
&\quad - H\left(X_i \mid X^{i-1}, Z_i^n, T_1, T_2, T_3\right) \quad (43)
\end{aligned}
$$

where equality (a) is due to the Markov chain $X_i - (X^{i-1}, Z_i^n, T_1, T_2) - Z^{i-1}$ (see Fig. 9). Now let us de-
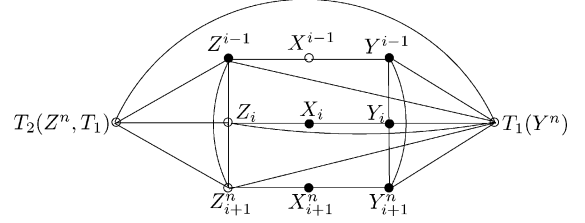

Fig. 9. Graphical proof of the Markov chain $X_i - (X^{i-1}, Z_i^n, T_1, T_2) - Z^{i-1}$. The undirected graph corresponds to the joint distribution $p(x^{i-1}, z^{i-1}) \, p(y^{i-1} \mid x^{i-1}) \, p(x_i, z_i) \, p(y_i \mid x_i) \, p(x_{i+1}^n, z_{i+1}^n) \, p(y_{i+1}^n \mid x_{i+1}^n) \, p(t_1 \mid y^n) \, p(t_2 \mid z^n, t_1)$. The Markov chain holds since all paths from Z^{i-1} to X_i pass through $(X^{i-1}, Z_i^n, T_1, T_2)$.

note $U_i \triangleq X^{i-1}, T_1, Z_{i+1}^n$, $V_i \triangleq T_2$ and $W_i \triangleq T_3$, and we obtain from (41)–(43)

$$R_1 \geq \frac{1}{n} \sum_{i=1}^{n} I(Y_i; U_i \mid Z_i)$$

$$R_2 \geq \frac{1}{n} \sum_{i=1}^{n} I(Z_i; V_i \mid U_i, X_i)$$

$$R_3 \geq \frac{1}{n} \sum_{i=1}^{n} I(X_i; W_i \mid U_i, V_i, Z_i). \quad (44)$$

Now, we verify that the joint distribution of $(X_i, Y_i, Z_i, U_i, V_i, W_i)$ is of the form (38), i.e., $U_i - Y_i - (Z_i, X_i)$, $V_i - (U_i, Z_i) - (Y_i, X_i)$ and $W_i - (U_i, V_i, X_i, Y_i) - Z_i$, hold. The Markov chain $(T_1(Y^n), X^{i-1}, Z_{i+1}^n) - Y_i - (Z_i, X_i)$ trivially holds, and the Markov chains

$$Z^{i-1} - \left(T_1(Y^n), X^{i-1}, Z_i^n\right) - (Y_i, X_i) \quad (45)$$
$$X_{i+1}^n - \left(T_1(Y^n), T_2(T_1, Z^n), X^i, Z_{i+1}^n, Y_i\right) - Z_i \quad (46)$$

are proven in is proven in Figs. 10 and 11, respectively.

Next, we show that there exist sequences of functions $\{\hat{Z}_i(U_i, V_i, Z_i)\}$, and $\{\hat{X}_i(U_i, W_i, Z_i)\}$ such that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{X}_i(U_i, W_i, Z_i))] \leq D_x$$

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{Z}_i(U_i, V_i, X_i))] \leq D_z. \quad (47)$$

The only difficulty here is that the terms in (U_i, W_i, Z_i) do not include Z^{i-1} and the terms (U_i, V_i, X_i) do not include X_{i+1}^n. However, this is solved by the same argument as for the Wyner–Ziv with helper at the end of Section V, by showing the Markov forms $X_i - (U_i, W_i, Z_i) - Z^{i-1}$ and $Z_i - (U_i, V_i, X_i) - X_{i+1}^n$ for which the proofs are given in Figs. 12 and 13, respectively.

Finally, let Q be a random variable independent of X^n, Y^n, Z^n, and uniformly distributed over the set $\{1, 2, 3, \ldots, n\}$. Define the random variables $U \triangleq (Q, U_Q)$, $V \triangleq (Q, V_Q)$, $W \triangleq (Q, W_Q)$, $\hat{X} \triangleq \hat{X}_Q$, and $\hat{Z} \triangleq \hat{Z}_Q$. Then (44)–(47) imply that the equations that define $\mathcal{R}(D_x, D_z)$ i.e., (1)–(5), hold. ∎

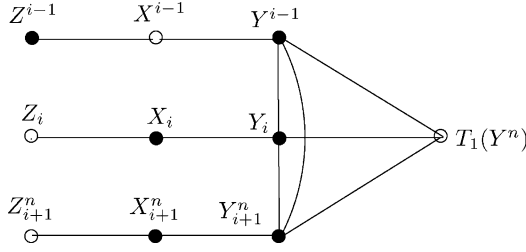Fig. 10. Graphical proof of the Markov chain Z^{i-1} — $(T_1(Y^n), X^{i-1}, Z_i^n)$ — (Y_i, X_i). The undirected graph corresponds to the joint distribution $p(x^{i-1}, z^{i-1}) p(y^{i-1} | x^{i-1}) p(x_i, z_i) p(y_i|x_i) p(x_{i+1}^n, z_{i+1}^n) p(y_{i+1}^n|x_{i+1}^n) p(t_1|y^n)$. The Markov chain holds since all paths from Z^{i-1} to (X_i, Y_i) pass through (X^{i-1}, Z_i^n, T_1).

Fig. 12. Graphical proof of the Markov chain $X_i - (U_i, W_i, X_i) - Z^{i-1}$. The undirected graph corresponds to the joint distribution $p(x^{i-1}, z^{i-1}) p(y^{i-1} | x^{i-1}) p(x_i, z_i) p(y_i | x_i) p(x_{i+1}^n, z_{i+1}^n) p(y_{i+1}^n | x_{i+1}^n) p(t_1 | y^n) p(t_3 | x^n, t_1)$. The Markov chain holds since all paths from X_i to Z^{i-1} pass through $(T_1(Y^n), T_3(T_1, X^n), X^{i-1}, Z_{i+1}^n)$.
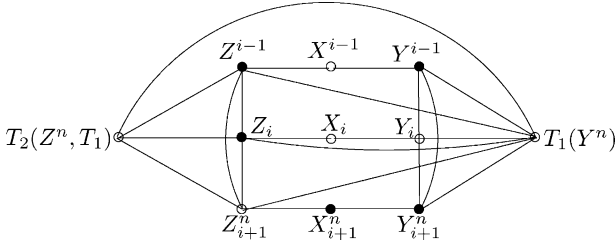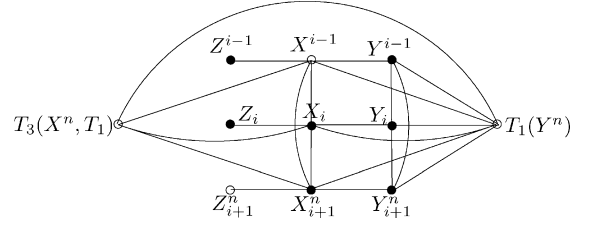

Fig. 11. Graphical proof of the Markov chain $X_{i+1}^n - (T_1(Y^n), T_2(T_1, Z^n), X^i, Z_{i+1}^n, Y_i) - Z_i$. The undirected graph corresponds to the joint distribution $p(x^{i-1}, y^{i-1}) p(z^{i-1} | y^{i-1}) p(x_i, y_i) p(z_i | y_i) p(x_{i+1}^n, y_{i+1}^n) p(z_{i+1}^n | y_{i+1}^n) p(t_1 | y^n) p(t_2 | z^n, t_1)$. The Markov chain holds since all paths from Z^i to X_{i+1}^n pass through $(T_1(Y^n), T_2(T_1, Z^n), X^i, Z_{i+1}^n, Y_i)$.
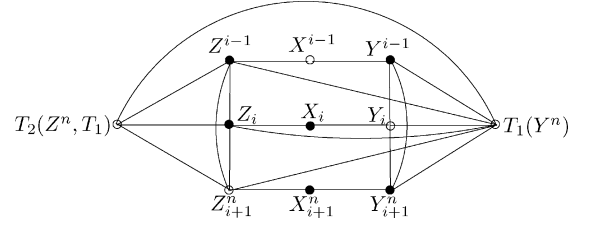

Fig. 13. Graphical proof of the Markov chain $Z_i - (T_1(Y^n), T_2(T_1, Z^n), X^{i-1}, Z_{i+1}^n) - X_{i+1}^n$. The undirected graph corresponds to the joint distribution $p(x^{i-1}, z^{i-1}) p(y^{i-1} | x^{i-1}) p(x_i, z_i) p(y_i | x_i) p(x_{i+1}^n, z_{i+1}^n) p(y_{i+1}^n | x_{i+1}^n) p(t_1 | y^n) p(t_2 | z^n, t_1)$. The Markov chain holds since all paths from Z^{i-1} to X_i pass through $(T_1(Y^n), T_2(T_1, Z^n), X^{i-1}, Z_{i+1}^n)$.

VII. TWO-WAY MULTI STAGE

Here we consider the two-way multistage rate-distortion problem with a helper. First, the helper sends a common message to both users, and then users X and Z send to each other a total rate R_x and R_z, respectively, in K rounds. We use the definition of two-way source coding as given in [1], where each user may transmit up to K messages to the other user that depends on the source and previous received messages.

Let $\mathcal{M}$ denote a set of positive integers $\{1, 2, \ldots, M\}$ and let $\mathcal{M}^K$ the collection of K sets $\{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_K\}$.

Definition 4: An $(n, M_y, M_x^K, M_z^K, D_x, D_z)$ code for two sources X and Z with helper Y consists of the encoders

$$f_y : \mathcal{Y}^n \to \mathcal{M}_y$$
$$f_{z,k} : \mathcal{Z}^n \times \mathcal{M}_x^{k-1} \times \mathcal{M}_y \to \mathcal{M}_{z,k}, \quad k = 1, 2, \ldots, K$$
$$f_{x,k} : \mathcal{X}^n \times \mathcal{M}_z^k \times \mathcal{M}_y \to \mathcal{M}_{x,k}, \quad k = 1, 2, \ldots, K \quad (48)$$

and two decoders

$$g_x : \mathcal{X}^n \times \mathcal{M}_y \times \mathcal{M}_z^K \to \hat{\mathcal{Z}}^n$$
$$g_z : \mathcal{Z}^n \times \mathcal{M}_y \times \mathcal{M}_x^K \to \hat{\mathcal{X}}^n \quad (49)$$

such that

$$\mathbb{E}\left[\sum_{i=1}^n d_x(X_i, \hat{X}_i)\right] \le D_x$$
$$\mathbb{E}\left[\sum_{i=1}^n d_z(Z_i, \hat{Z}_i)\right] \le D_z. \quad (50)$$

The rate triple (R_x, R_y, R_z) of the code is defined by

$$R_y = \frac{1}{n} \log M_y;$$
$$R_x = \frac{1}{n} \sum_{i=1}^K \log M_{x,i};$$
$$R_z = \frac{1}{n} \sum_{i=1}^K \log M_{z,i}; \quad (51)$$

Let us denote by $\mathcal{R}_K^O(D_x, D_z)$ the (operational) achievable region of the multistage rate distortion with a helper, i.e., the closure of the set of all triple rate (R_x, R_y, R_z) that are achievable with a distortion pair (D_x, D_z). Let $\mathcal{R}_K(D_x, D_z)$ be the set of all triple rates (R_x, R_y, R_z) that satisfy

$$R_y \ge I(U; Y) \quad (52)$$
$$R_z \ge \sum_{k=1}^K I(Z; V_k \,|\, X, U, V^{k-1}, W^{k-1}) \quad (53)$$
$$R_x \ge \sum_{k=1}^K I(X; W_k \,|\, Z, U, V^k, W^{k-1}) \quad (54)$$

for some auxiliary random variables (U, V^K, W^k) that satisfy

$$U - Y - (X, Z) \quad (55)$$
$$V_k - (Z, U, V^{k-1}, W^{k-1}) - (X, Y)$$
$$\qquad\qquad k = 1, 2, \ldots, K, \quad (56)$$
$$W_k - (X, U, V^k, W^{k-1}) - (Z, Y)$$
$$\qquad\qquad k = 1, 2, \ldots, K, \quad (57)$$
$$\mathbb{E}d_x(X, \hat{X}(U, W^K, Z)) \le D_x$$
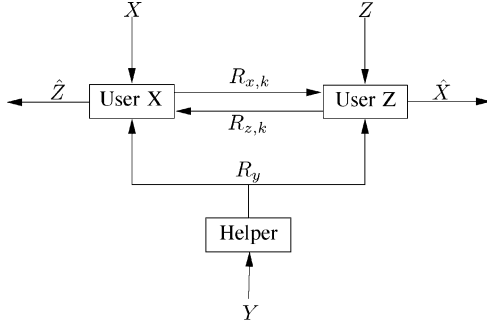$$\mathbb{E}d_z(Z, \hat{Z}(U, V^K, X)) \le D_z. \quad (58)$$

Fig. 14. Two-way multistage with a helper. First Helper Y sends a common message to User X and to User Z at rate R_y, and then we have K rounds where in each round $k \in \{1, \ldots, K\}$ User Z sends a message to User X at rate $R_{z,k}$, and User X sends a message to User Z at rate $R_{x,k}$. The limitation is on rate R_y and on the sum rates $R_x = \sum_{k=1}^{K} R_{x,k}$ and $R_z = \sum_{k=1}^{K} R_{z,k}$. We assume that the side information Y and the two sources X, Z are i.i.d. and form the Markov chain $Y - X - Z$.

The Markov chain $Y - X - Z$ and the Markov chains given in (55)–(57) imply that the joint distribution of X, Y, Z, U, V^k, W^k is of the form $p(x,y)p(z \,|\, x)p(u \,|\, y) \prod_{k=1}^{K} p(v_k \,|\, z, u, v^{k-1}, w^{k-1}) \ p(w_k \,|\, x, u, v^k, w^{k-1})$. Furthermore, (53) and (54) can be written as

$$R_z \geq I(Z; V^K, W^K \,|\, X, U) \qquad (59)$$
$$R_x \geq I(X; V^K, W^K \,|\, Z, U) \qquad (60)$$

due to the Markov chains $Z - (X, U, V^k, W^{k-1}) - W_k$ and $X - (Z, U, V^{k-1}, W^{k-1}) - V_k$.

Lemma 9:
1) The region $\mathcal{R}_K(D_x, D_z)$ is convex.
2) To exhaust $\mathcal{R}_K(D_x, D_z)$, it is enough to restrict the alphabet of U, V, and W to satisfy

$$|\mathcal{U}| \leq |\mathcal{Y}| + 2K + 1$$
$$|\mathcal{V}_k| \leq |\mathcal{Z}||\mathcal{U}||\mathcal{V}^{k-1}||\mathcal{W}^{k-1}| + 2(K + 1 - k) + 1$$
$$\text{for } k = 1, \ldots, K$$
$$|\mathcal{W}_k| \leq |\mathcal{X}||\mathcal{U}||\mathcal{V}^k||\mathcal{W}^{k-1}| + 2(K + 1 - k)$$
$$\text{for } k = 1, \ldots, K. \qquad (61)$$

The proof of the lemma is analogous to the proof of Lemma 2 and, therefore, omitted.

Theorem 10: In the two-way problem with K stages of communication and a helper, as depicted in Fig. 14, where $Y - X - Z$

$$\mathcal{R}_K^O(D_x, D_z) = \mathcal{R}_K(D_x, D_z). \qquad (62)$$

Theorem 10 is a generalization of Theorem 1 [(52)–(58) where $K = 1$ are equivalent to (1)–(5)] and its proof is a straightforward extension. Here we explain only the extensions.

Sketch of achievability: In the achievability proof of Theorem 1, we generated the sequences (U^n, V_1^n, W_1^n) that are jointly typical with X^n, Y^n, Z^n. Using the same idea of Wyner–Ziv coding we continue and generate at any stage $k = 1, 2, \ldots, K$, the sequence V_k^n that is jointly typical with the other sequences by transmitting a message at rate $I(Z; V_k | X, U, V^{k-1}, W^{k-1})$ from User Z to User X, and similarly the sequence W_k^n that is jointly typical with the other sequences by transmitting a message at rate

$I(X; W_k \,|\, Z, U, V^k, W^{k-1})$ from User X to User Z. In the final stage, User X uses the sequences $(X^n, U^n, V_1^n, \ldots, V_K^n)$ to construct $\hat{Z}^n$ and, similarly, User Z uses the sequences $(Z^n, U^n, W_1^n, \ldots, W_K^n)$ to construct $\hat{X}^n$.

Sketch of Converse: Assume that we have an $(n, M_y, M_x^K, M_z^K, D_x, D_z)$ code and we will show the existence of a vector $(U, V^K, W^K, \hat{X}, \hat{Z})$ that satisfy (52)–(58). Denote $T_y = f_y(Y^n)$, $T_{z,k} = f_{z,k}(Z^n, T_y, T_x^{k-1})$, and $T_{x,k} = f_{x,k}(X^n, T_y, T_z^k)$. Then the same arguments as in (41) we obtain

$$nR_y \geq \sum_{i=1}^{n} H\left(Y_i; X^{i-1}, T_y, Z_{i+1}^n \,\middle|\, Z_i\right). \qquad (63)$$

Then we have

$$nR_z \geq H\left(T_z^K\right) = \sum_{k=1}^{K} H\left(T_{z,k} \,\middle|\, T_z^{k-1}\right)$$
$$\geq \sum_{k=1}^{K} H\left(T_{z,k} \,\middle|\, T_z^{k-1}, T_x^{k-1}\right) \qquad (64)$$
$$nR_x \geq H\left(T_x^K\right) = \sum_{k=1}^{K} H\left(T_{x,k} \,\middle|\, T_x^{k-1}\right)$$
$$\geq \sum_{k=1}^{K} H\left(T_{x,k} \,\middle|\, T_x^{k-1}, T_z^k\right). \qquad (65)$$

Applying the same arguments as in (42) and (43) on the terms in (64) and (65), respectively, we obtain

$$H\left(T_{z,k} \,\middle|\, T_z^{k-1}, T_x^{k-1}\right)$$
$$\geq \sum_{i=1}^{n} I\left(Z_i; T_{z,k} \,\middle|\, Z_{i+1}^n, X^i, T_y, T_z^{k-1}, T_x^{k-1}\right)$$
$$H\left(T_{x,k} \,\middle|\, T_x^{k-1}, T_z^k\right)$$
$$\geq \sum_{i=1}^{n} I\left(X_i; T_{x,k} \,\middle|\, Z_i^n, X^{i-1}, T_y, T_z^k, T_x^{k-1}\right). \quad (66)$$

We define the auxiliary random variables as $U \triangleq X^{Q-1}, T_y, Z_{Q+1}^n$, $V_k = T_{z,k}$ and $W_k = T_{x,k}$, where Q is distributed uniformly on the integers $\{1, 2, \ldots, n\}$ (independent of all else). ∎

VIII. GAUSSIAN CASE

In this section we consider the Gaussian instance of the two way setting with a helper as defined in Section III and explicitly express the region for a mean square error distortion.

Since X, Y, Z form the Markov chain $Y - X - Z$, we assume, without loss of generality, that $X = Z + A$ and $Y = Z + A + B$, where the random variables (A, B, Z) are zero-mean Gaussian and independent of each other, where $\mathbb{E}[A^2] = \sigma_A^2$, $\mathbb{E}[B^2] = \sigma_B^2$ and $\mathbb{E}[Z^2] = \sigma_Z^2$.

Corollary 11: The achievable rate region of the problem illustrated in Fig. 15 is

$$R_z \geq \frac{1}{2} \log \frac{\sigma_A^2 \sigma_Z^2}{D_z \left(\sigma_A^2 + \sigma_Z^2\right)} \qquad (67)$$
$$R_x \geq \frac{1}{2} \log \frac{\sigma_A^2 \left(\sigma_B^2 + \sigma_A^2 2^{-2R_y}\right)}{D_x \left(\sigma_A^2 + \sigma_B^2\right)} \qquad (68)$$
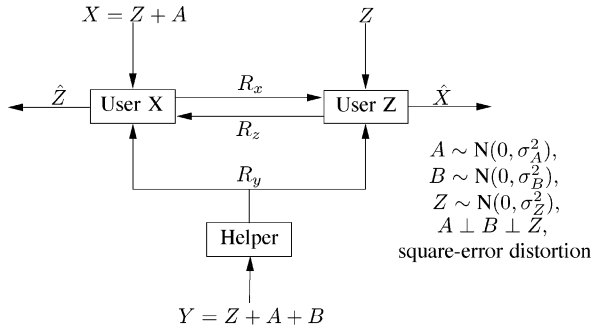
Fig. 15. Gaussian two-way with a helper. The side information Y and the two sources X, Z are i.i.d., jointly Gaussian and form the Markov chain $Y - X - Z$. The distortion is the square error, i.e., $d_x(X^n, \hat{X}^n) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{X}_i)^2$ and $d_z(Z^n, \hat{Z}^n) = \frac{1}{n}\sum_{i=1}^{n}(Z_i - \hat{Z}_i)^2$.
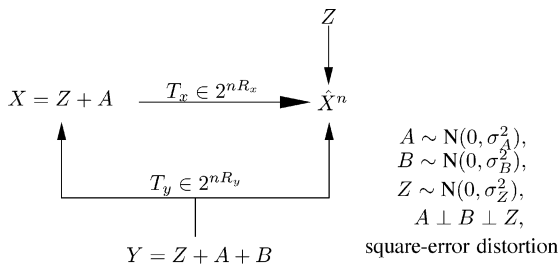

Fig. 16. Gaussian case: the zero-mean Gaussian random variables A, B, Z are i.i.d. and independent of each other. Their variances are σ_A^2, σ_B^2 and σ_Z^2, respectively. The source X and the helper Y satisfy $X = A + Z$ and $Y = Z + A + B$. The distortion is the square error, i.e., $d(X^n, \hat{X}^n) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{X}_i)^2$.

and the multi stage option as presented in Section VII does not increase the rate region for quadratic Gaussian case.

Proof: The converse and achievability of (67) follows from the Gaussian Wyner–Ziv coding [20] result, which states that the achievable rate for the Gaussian Wyner–Ziv setting is the same as the case where the side information is known to the encoder and decoder. Namely, even if X is known to User Z, the rate R_z won't change; this also implies that a multi stage setting wont influence the total rate R_z that is achievable and hence the whole achievable region. Furthermore, because of the Markov chain $Z - X - Y$, the rate R_y does not have any influence on R_z, since this rate is the achievable rate even if Y is known to both users. The achievability and the converse for R_x is given in the following corollary. ∎

Corollary 12: The achievable rate region of the problem illustrated in Fig. 16 is

$$R \geq \frac{1}{2} \log \frac{\sigma_A^2 \left(1 - \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2}(1 - 2^{-2R_y})\right)}{D}. \quad (69)$$

It is interesting to note that the rate region does not depend on σ_Z^2. Furthermore, we show in the proof that for the Gaussian case the rate region is the same as when Z is known to the source X and the helper Y.

Proof of Corollary 12: **Converse**: Assume that both encoders observe Z^n. Without loss of generality, the encoders can subtract Z from X and Y; hence the problem is equivalent to

new rate distortion problem with a helper, where the source is A and the helper is $A + B$. Now using the result for the Gaussian case from [8], adapted to our notation, we obtain (69).

Achievability: Before proving the direct-part of Corollary 12, we establish the following lemma which is proved in Appendix C.

Lemma 13: (Gaussian Wyner–Ziv rate-distortion problem with additional side information known to the encoder and decoder). Let (X, W, Z) be jointly Gaussian. Consider the Wyner–Ziv rate distortion problem where the source X is to be compressed with quadratic distortion measure, W is available at the encoder and decoder, and Z is available only at the decoder. The rate-distortion region for this problem is given by

$$R(D) = \frac{1}{2} \log \frac{\sigma_{X|W,Z}^2}{D} \quad (70)$$

where $\sigma_{X|W,Z}^2 = \mathbb{E}[(X - \mathbb{E}[X \,|\, W, Z])^2]$, i.e., the minimum square error of estimating X from (W, Z).

Let $V = A + B + Z + D$, where $D \sim \mathrm{N}(0, \sigma_D^2)$ and is independent of (A, B, Z). Clearly, we have $V - Y - X - Z$. Now, let us generate V at the source-encoder and at the decoder using the achievability scheme of Wyner [20]. Since $I(V; Z) \leq I(V; X)$ a rate $R' = I(V; Y) - I(V; Z)$ would suffice, and it may be expressed as follows:

$$\begin{aligned}
R' &= I(V; Y \,|\, Z) \\
&= h(V \,|\, Z) - h(V \,|\, Y) \\
&= \frac{1}{2} \log \frac{\sigma_A^2 + \sigma_B^2 + \sigma_D^2}{\sigma_D^2} \quad (71)
\end{aligned}$$

and this implies that

$$\sigma_D^2 = \frac{\sigma_A^2 + \sigma_B^2}{2^{2R'} - 1}. \quad (72)$$

Now, we invoke Lemma 13, where V is the side information known both to the encoder and decoder; hence, a rate that satisfies the following inequality achieves a distortion D

$$\begin{aligned}
R &\geq \frac{1}{2} \log \frac{\sigma_{X|V,Z}^2}{D} \\
&= \frac{1}{2} \log \frac{\sigma_A^2}{D} \left(1 - \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2 + \sigma_D^2}\right). \quad (73)
\end{aligned}$$

Finally, by replacing σ_D^2 with the identity in (72) we obtain (69). ∎

IX. FURTHER RESULTS ON WYNER–ZIV WITH A HELPER WHERE $Y - X - Z$

In this section we investigate two properties of the rate-region of the Wyner–Ziv setting (Fig. 17) with a Markov form $Y - X - Z$. First, we investigate the tradeoff between the rate sent by the helper and the rate sent by the source and roughly speaking we conclude that a bit from the source is more "valuable" than a bit from the helper. Second, we examine the case where the helper has the freedom to send different messages, at different rates, to the encoder and the decoder. We show that "more help" to the encoder than to the decoder does not yield any performance gain
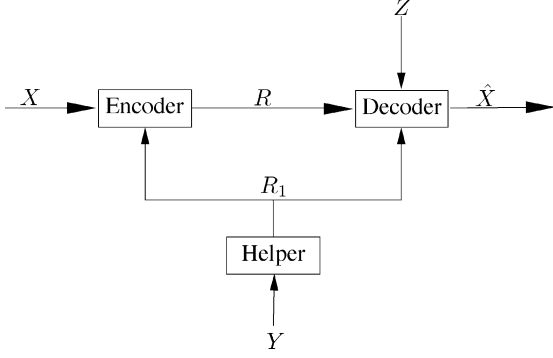
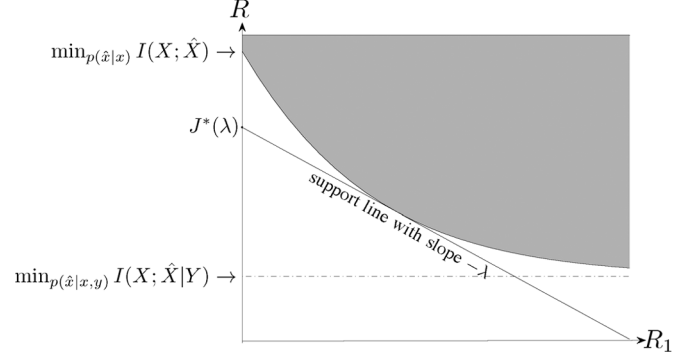Fig. 17. Wyner–Ziv problem with a helper where the Markov chain $Y - X - Z$ holds.

Fig. 18. Support line of $R(R_1)$ with a slope $-\lambda$. $J*(\lambda)$ is the intersection of the support line with the R axis.

and that in such cases the freedom to send different messages to the encoder and the decoder yields no gain over the case of a common message. Further, in this setting of different messages, the rate to the encoder can be strictly less than that to the decoder with no performance loss.

A. Bit From the Source-Encoder versus a Bit From the Helper

Assume that we have a sequence of $(n, 2^{nR}, 2^{nR_1})$ codes that achieves a distortion D, such that the triple (R, R_1, D) is on the border of the region $\mathcal{R}_{Y-X-Z}(D)$ [recall the definition of $\mathcal{R}_{Y-X-Z}(D)$ in (15)–(17)]. Now, suppose that the helper is allowed to increase the rate by an amount $\Delta' > 0$ to $R_1 + \Delta'$; to what rate $R - \Delta$ can the source-encoder reduce its rate and achieve the same distortion D?

Despite the fact that the additional rate Δ' is transmitted both to the decoder and encoder, we show that always $\Delta \leq \Delta'$. Let us denote by $R(R_1)$ the boundary of the region $\mathcal{R}_{Y-X-Z}(D)$ for a fixed D. We formally show that $\Delta \leq \Delta'$ by proving that the slope of the curve $R(R_1)$ is always less than 1. The proof uses similar technique as in [21].

Lemma 14: For any $Y - X - Z$, D, and R_1, the subgradients of the curve $R(R_1)$ are less than 1.

Proof: Since $\mathcal{R}_{Y-X-Z}(D)$ is a convex set, $R(R_1)$ is a convex function. Furthermore, $R(R_1)$ is nonincreasing in R_1. Now, let us define $J^*(\lambda)$ as

$$J^*(\lambda) = \min_{p(x,y,z,u,w) \in \mathcal{P}} I(X; W \mid U, Z) + \lambda I(Y; U \mid Z) \tag{74}$$

where $\mathcal{P}$ is the set of distributions satisfying $p(x, y, z, u, w, \hat{x}) = p(x, y)p(z \mid y) p(u \mid y) p(w \mid u, x)p(\hat{x} \mid u, w, z)$, $\mathbb{E}d(X, \hat{X}) \leq D$. The line $J^*(\lambda) = R + \lambda R$ is a support line of $R(\bar{R_1})$, and therefore, λ is a subgradient. The value $J^*(\lambda)$ is the intersection between the support line with slope $-\lambda$ and the axis R, as shown in Fig. 18. Because of the convexity and the monotonicity of $R(R_1)$, $J^*(\lambda)$ is upper-bounded by $R(0)$, i.e.

$$J^*(\lambda) \leq \min_{p(\hat{x},x,y,z,u,w) \in \mathcal{P}} R(0)$$
$$= \min_{p(\hat{x},x,y,z,w) \in \mathcal{P}_{WZ}} I(X; W \mid Z) \tag{75}$$

where $\mathcal{P}_{WZ}$ is the set of distributions that satisfies $p(\hat{x}, x, z, w) = p(x)p(z \mid x)p(w \mid x)p(\hat{x} \mid w, z)$, $\mathbb{E}d(X, \hat{X}) \leq D$.
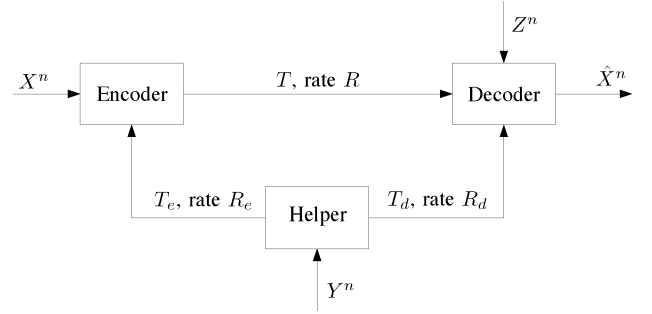

Fig. 19. Rate distortion problem with decoder side information, and independent helper rates. We assume the Markov relation $Y - X - Z$.

In addition, we observe that

$$J^*(1) = \min_{p(x,y,z,u,w,\hat{x}) \in \mathcal{P}} I(X; W \mid U, Z) + I(Y; U \mid Z)$$
$$\overset{(a)}{=} \min_{p(x,y,z,u,w,\hat{x}) \in \mathcal{P}} I(X, Y; W, U \mid Z)$$
$$\geq \min_{p(x,y,z,u,w,\hat{x}) \in \mathcal{P}} I(X; W \mid Z)$$
$$= \min_{p(\hat{x},x,y,z,w) \in \mathcal{P}_{WZ}} I(X; W \mid Z) \tag{76}$$

where step (a) is due to the Markov chains $U - Y - (Z, X)$ and $W - (U, X) - (Y, Z)$. Combining (75) and (76), we conclude that for any subgradient $-\lambda$, $J^*(\lambda) \leq J^*(1)$. Since $J^*(\lambda)$ is increasing in λ, we conclude that $\lambda \leq 1$.   ∎

An alternative and equivalent proof would be to claim that, since $R(R_1)$ is a convex and non increasing function, $\frac{\Delta}{\Delta'} \leq |\frac{dR}{dR_1}|_{R_1=0}$, and then to claim that the largest slope at $R_1 = 0$ is when $Y = X$, which is 1. For the Gaussian case, the derivative may be calculated explicitly from (69), in particular for $R_1 = 0$, and we obtain

$$\Delta \leq \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2} \Delta'. \tag{77}$$

B. Case of Independent Rates

Here, we treat the rate distortion scenario where side information from the helper is encoded using two different messages, possibly at different rates, one to the encoder and one to the decoder, as shown in Fig. 19. The complete characterization of achievable rates for this scenario is still an open problem. However, the solution that is given in previous sections, where there is one message known both to the encoder and decoder, provides

us insight that allows us to solve several cases of the problem shown here. We start with the definition of the general case.

Definition 5: An (n, M, M_e, M_d, D) code for source X with side information Y and different helper messages to the encoder and decoder, consists of three encoders

$$f_e : \mathcal{Y}^n \to \{1, 2, \ldots, M_e\}$$
$$f_d : \mathcal{Y}^n \to \{1, 2, \ldots, M_d\}$$
$$f : \mathcal{X}^n \times \{1, 2, \ldots, M_e\} \to \{1, 2, \ldots, M\} \quad (78)$$

and a decoder

$$g : \{1, 2, \ldots, M\} \times \{1, 2, \ldots, M_d\} \to \hat{\mathcal{X}}^n \quad (79)$$

such that

$$\mathbb{E}d(X^n, \hat{X}^n) \leq D. \quad (80)$$

To avoid cumbersome statements, we will not repeat in the sequel the words "... different helper messages to the encoder and decoder," as this is the topic of this section, and should be clear from the context. The rate pair (R, R_e, R_d) of the (n, M, M_e, M_d, D) code is

$$R = \frac{1}{n} \log M$$
$$R_e = \frac{1}{n} \log M_e$$
$$R_d = \frac{1}{n} \log M_d \quad (81)$$

Definition 6: Given a distortion D, a rate triple (R, R_e, R_d) is said to be *achievable* if for any $\delta > 0$, and sufficiently large n, there exists an $(n, 2^{n(R+\delta)}, 2^{n(R_e+\delta)}, 2^{n(R_d+\delta)}, D+\delta)$ code for the source X with side information Y.

Definition 7: The (operational) achievable region $\mathcal{R}_g^O(D)$ *of rate distortion with a helper known at the encoder and decoder is the closure of the set of all achievable rate triples at distortion* D.

Denote by $\mathcal{R}_g^O(R_e, R_d, D)$ the *section* of $\mathcal{R}_g^O(D)$ at helper rates (R_e, R_d). That is

$$\mathcal{R}_g^O(R_e, R_d, D) = \{R : (R, R_e, R_d)$$
$$\text{are achievable with distortion } D\} \quad (82)$$

and similarly, denote by $\mathcal{R}(R_1, D)$ the section of the region $\mathcal{R}_{Y-X-Z}(D)$, defined in (15)–(18) at helper rate R_1. Recall that, according to Theorem 4, $\mathcal{R}(R_1, D)$ consists of all achievable source coding rates when the helper sends common messages to the source encoder and destination at rate R_1. The main result of this section is the following.

Theorem 15: For any $R_e \geq R_d$

$$\mathcal{R}_g^O(R_e, R_d, D) = \mathcal{R}(R_d, D). \quad (83)$$

Theorem 15 has interesting implications on the coding strategy taken by the helper. It says that no gain in performance can be achieved if the source encoder gets "more help" than the

decoder (i.e., if $R_e > R_d$), and thus we may restrict R_e to be no higher than R_d. Moreover, in those cases where $R_e = R_d$, optimal performance is achieved when the helper sends to the encoder and decoder exactly the same message. The proof of this statement uses operational arguments.

Proof of Theorem 15: Clearly, the claim is proved once we show the statement for $R_e = H(Y)$. In this situation, we can equally well assume that the encoder has full access to Y. Thus, fix a general scheme like in Definition 5 with $R_e = H(Y)$. The encoder is a function of the form $f(X^n, Y^n)$. Define $T_2 = f_d(Y^n)$. The Markov chain $Z - X - Y$ implies that $Z^n - (X^n, T_2) - Y^n$ also forms a Markov chain. This implies, in turn that there exists a function ϕ and a random variable W, uniformly distributed in $[0, 1]$ and independent of (X^n, T_2, Z^n), such that

$$Y^n = \phi(X^n, T_2, W). \quad (84)$$

Thus the source encoder operation can be written as

$$f(X^n, Y^n) = f(X^n, \phi(X^n, T_2, W))$$
$$\triangleq \tilde{f}(X^n, T_2, W) \quad (85)$$

implying, in turn, that the distortion of this scheme can be expressed as

$$d(X^n, \hat{X}^n)$$
$$= \mathbb{E}[d(X^n, \hat{X}^n(\tilde{f}(X^n, T_2, W), T_2, Z^n))]$$
$$\stackrel{(a)}{=} \int_0^1 \mathbb{E}[d(X^n, \hat{X}^n(\tilde{f}(X^n, T_2, w), T_2, Z^n))]dw$$
$$\stackrel{(b)}{=} \int_0^1 \mathbb{E}[d(X^n, \hat{X}^n(f^w(X^n, T_2), T_2, Z^n))]dw \quad (86)$$

where (a) holds since W is independent of (X^n, T_2, Z^n), and (b) by defining

$$f^w(X^n, T_2) = \tilde{f}(X^n, T_2, w). \quad (87)$$

Note that for a given w, the function f^w is of the form of encoding functions where the helper sends one message to the encoder and decoder. Therefore, we conclude that anything achievable with a scheme from Definition 5, is achievable by time-sharing where the helper sends one message to the encoder and decoder. ∎

The statement of Theorem 15 can be extended to rates R_e slightly lower than R_d. This extension is based on the simple observation that the source encoder knows X, which can serve as side information in decoding the message sent by the helper. Therefore, any message T_2 sent to the source decoder can undergo a stage of binning with respect to X. As an extreme example, consider the case where $R_e \geq H(Y \mid X)$. The source encoder can fully recover Y, hence, there is no advantage in transmitting to the encoder at rates higher than $H(Y \mid X)$; the decoder, on the other hand, can benefit from rates in the region $H(Y \mid X) < R_d < H(Y \mid Z)$. This rate interval is not empty due to the Markov chain $Y - X - Z$. These observations are summarized in the next theorem.

Theorem 16:

1) Let (U, V) achieve a point (R, R') in $\mathcal{R}_{Y-X-Z}(D)$, i.e.,

$$R = I(X; U \mid V, Z)$$

$$R' = I(Y; V \mid Z) = I(V; Y) - I(V; Z) \qquad (88)$$

$$D \geq \mathbb{E}d(X, \hat{X}(U, V, Z)) \qquad (89)$$

$$V - Y - X - Z. \qquad (90)$$

Then $(R, R_e, R') \in \mathcal{R}_g^O(D)$ for every R_e satisfying

$$R_e \geq I(V; Y \mid Z) - I(V; X \mid Z)$$

$$= I(V; Y) - I(V; X). \qquad (91)$$

2) Let (R, R') be an outer point of $\mathcal{R}_{Y-X-Z}(D)$. That is

$$(R, R') \notin \mathcal{R}_{Y-X-Z}(D). \qquad (92)$$

Then (R, R_e, R') is an outer point of $\mathcal{R}_g^O(D)$ for any R_e, i.e.,

$$(R, R_e, R') \notin \mathcal{R}_g^O(D) \quad \forall R_e. \qquad (93)$$

The proof of Part 1 is based on binning, as described above. In particular, observe that R_e given in (91) is lower than R' of (88) due to the Markov chain $V - Y - X - Z$. Part 2 is a partial converse, and is a direct consequence of Theorem 15. The details, being straightforward, are omitted.

APPENDIX A
PROOF OF THE TECHNIQUE FOR VERIFYING MARKOV RELATIONS

Proof: First let us prove that three random variables X, Y, Z, with a joint distribution of the form

$$p(x, y, z) = f(x, y)f(y, z) \qquad (94)$$

satisfy the Markov chain $Y - X - Z$. Consider

$$p(z|y, x) = \frac{f(x, y)f(y, z)}{f(x, y)\left(\sum_z f(y, z)\right)} = \frac{f(y, z)}{\sum_z f(y, z)} \qquad (95)$$

and since the expression does not include the argument x we conclude that $p(z \mid y, x) = p(z \mid y)$.

For the more general case, we first extend the sets $X_{\mathcal{G}_1}$ $X_{\mathcal{G}_3}$. We start by defining $\bar{\mathcal{G}}_1 = \mathcal{G}_1$ and $\bar{\mathcal{G}}_3 = \mathcal{G}_3$, and then we add to $X_{\bar{\mathcal{G}}_1}$ and to $X_{\bar{\mathcal{G}}_3}$ all their neighbors that are not in $X_{\mathcal{G}_2}$ (a neighbor to a group is a node that is connected by one edge to the an element in the group). We repeat this procedure till there are no more nodes to add to $X_{\bar{\mathcal{G}}_1}$ or $X_{\bar{\mathcal{G}}_3}$. Note that since there are no paths from $X_{\mathcal{G}_1}$ to $X_{\mathcal{G}_3}$ that do not pass through $X_{\mathcal{G}_2}$, then a node can not be added to both sets $X_{\bar{\mathcal{G}}_1}$ and $X_{\bar{\mathcal{G}}_3}$. The set of nodes that are not in $(X_{\bar{\mathcal{G}}_1}, X_{\mathcal{G}_2}, X_{\bar{\mathcal{G}}_3})$ is denoted as $X_{\mathcal{G}_0}$.

The sets $X_{\mathcal{G}_0}$ and $X_{\bar{\mathcal{G}}_1}$ and $X_{\bar{\mathcal{G}}_3}$ are connected only to $X_{\mathcal{G}_2}$ and not to each other, hence the joint distribution of $(X_{\mathcal{G}_0}, X_{\bar{\mathcal{G}}_1}, X_{\mathcal{G}_2}, X_{\bar{\mathcal{G}}_3})$ is of the following form:

$$p\left(X_{\mathcal{G}_0}, X_{\bar{\mathcal{G}}_1}, X_{\mathcal{G}_2}, X_{\bar{\mathcal{G}}_1}\right)$$
$$= f\left(X_{\mathcal{G}_0}, X_{\mathcal{G}_2}\right) f\left(X_{\bar{\mathcal{G}}_1}, X_{\mathcal{G}_2}\right) f\left(X_{\bar{\mathcal{G}}_3}, X_{\mathcal{G}_2}\right). \qquad (96)$$

By marginalizing over $X_{\mathcal{G}_0}$ and using the claim introduced in the first sentence of the proof we obtain the Markov chain $X_{\bar{\mathcal{G}}_1} - X_{\mathcal{G}_2} - X_{\bar{\mathcal{G}}_3}$, whcih implies $X_{\mathcal{G}_1} - X_{\mathcal{G}_2} - X_{\mathcal{G}_3}$. ∎

APPENDIX B
PROOF OF LEMMA 2

Proof: To prove Part 1, let Q be a time sharing random variable, independent of the source triple (X, Y, Z). Note that

$$I(Y; U|Z, Q) \overset{(a)}{=} I(Y; U, Q \mid Z) = I(Y; \tilde{U} \mid Z)$$
$$I(Z; V \mid U, X, Q) = I(Z; V \mid \tilde{U}, X)$$
$$I(X; W \mid U, V, Z, Q) = I(X; W \mid \tilde{U}, V, Z)$$

where $\tilde{U} = (U, Q)$, and in step (a) we used the fact that Y is independent of Q. This proves the convexity.

To prove Part 2, we invoke the support lemma [22, p. 310] three times, each time for one of the auxiliary random variables U, V, W. The external random variable U must have $|\mathcal{Y}| - 1$ letters to preserve $p(y)$ plus five more to preserve the expressions $I(Y; U \mid Z)$, $I(Z; V \mid U, X)$, $I(X; W \mid U, V, Z)$ and the distortions $\mathbb{E}d_x(X, \hat{X}(U, V, Z))$ $\mathbb{E}d_z(Z, \hat{Z}(U, W, X))$. Note that the joint $p(x, y, z)$ is preserved because of the Markov form $U - Y - X - Z$, and the structure of the joint distribution given in (4) does not change. We fix U, which now has a bounded cardinality, and we apply the support lemma for bounding V. The external random variable V must have $|\mathcal{U}||\mathcal{Z}| - 1$ letters to preserve $p(u, z)$ plus four more to preserve the expressions $I(Z; V \mid U, X)$, $I(X; W \mid U, V, Z)$ and the distortions $\mathbb{E}d_x(X, \hat{X}(U, V, Z))$, $\mathbb{E}d_z(Z, \hat{Z}(U, W, X))$. Note that because of the Markov structure $V - (U, Z) - (X, Y)$ the joint distribution $p(u, z, x, y)$ does not change. Finally, we fix U, V which now have a bounded cardinality and we apply the support lemma for bounding W. The external random variable W must have $|\mathcal{U}||\mathcal{V}||\mathcal{X}| - 1$ letters to preserve $p(u, v, x)$ plus two more to preserve the expressions $I(X; W \mid U, V, Z)$ and the distortions $\mathbb{E}d_z(Z, \hat{Z}(U, W, X))$. Note that because of the Markov structure $W - (U, V, X) - (Z, Y)$ the joint distribution $p(u, v, x, y, z)$ does not change. ∎

APPENDIX C
PROOF OF LEMMA 13

Since W, X, Z are jointly Gaussian, we have $\mathbb{E}[X \mid W, Z] = \alpha W + \beta Z$, for some scalars α, β. Furthermore, we have

$$X = \alpha W + \beta Z + N \qquad (97)$$

where N is a Gaussian random variable independent of (W, Z) with zero mean and variance $\sigma^2_{X \mid W, Z}$. Since W is known to the encoder and decoder we can subtract αW from X, and then using Wyner–Ziv coding for the Gaussian case [20] we obtain

$$R(D) = \frac{1}{2} \log \frac{\sigma^2_{X \mid W, Z}}{D}. \qquad (98)$$

Obviously, one can not achieve a rate smaller than this even if Z is known both to the encoder and decoder, and therefore this is the achievable region.

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