Network Coding Schemes for Data Exchange Networks With Arbitrary Transmission Delays

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***Abstract*— In this paper, we introduce construction techniques for network coding in bidirectional networks with arbitrary transmission delays. These coding schemes reduce the number of transmissions and achieve the optimal rate region in the corresponding broadcast model for both multiple unicast and multicast cases with up to three users, under the equal rate constraint. The coding schemes are presented in two phases; first, coding schemes for line, star and line-star topologies with arbitrary transmission delays are provided and second, any general topology with multiple bidirectional unicast and multicast sessions is shown to be decomposable into these canonical topologies to reduce the number of transmissions. As a result, the coding schemes developed for the line, star, and line-star topologies serve as building blocks for the construction of more general coding schemes for all networks. The proposed schemes are proved to be real time in the sense that they achieve the minimum decoding delay. With a negligible size header, these coding schemes are shown to be applicable to unsynchronized networks, i.e., networks with arbitrary transmission delays. Finally, we demonstrate the applicability of these schemes by extensive simulations. The implementation of such coding schemes on a wireless network with arbitrary transmission delays can improve performance and power efficiency.**

***Index Terms*— Multiple unicast, multicast, network coding, wireless networks, arbitrary delay.**

I. Introduction

Network Coding (NC) [1], [2] is a networking technique used to better exploit the available bandwidth [3], use energy efficiently [4] and increase the network's security [5]. Unlike the traditional approach of store-and-forward, NC enables messages to be encoded at the intermediate nodes. Yet, in many practical scenarios, arbitrary transmission delays may render existing NC solutions impractical.

The following example demonstrates the difficulty of implementing an NC scheme for a network with arbitrary transmission delays. Consider two users, 1 and 4, that exchange messages $\{W_1^{(0)}, W_1^{(1)}, \ldots, W_1^{(t)}\}$ and $\{W_4^{(0)}, W_4^{(1)}, \ldots, W_4^{(t)}\}$ through nodes 2 and 3 in a packet-based communication scheme (Fig. 1a), where the index t refers to the transmission time unit. The links can carry one packet per time unit, and each transmission has an integer arbitrary delay bounded by D. To achieve a minimum number of transmissions, we exploit NC and assume broadcast transmission, i.e., the same packet is transmitted to all adjacent nodes in each transmission; such an assumption is common in several channel models, e.g., wireless. While random linear network coding (RLNC) is capacity-achieving and the preferred solution in numerous scenarios, this is not the case here, and more structured codes are required. In general, RLNC[1] codes over all packets received on all incoming links to a node, so that packets will be linearly mixed together. This requires nodes to wait for all delayed packets to arrive, which causes unnecessary delay. Of course, the problem becomes even more involved with the inclusion of multiple sources and terminals, a scenario that introduces several independent, bi-directional sessions.

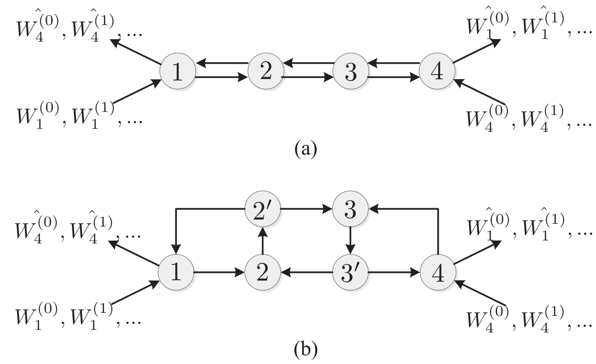

Fig. 1. Schematic illustrations of a network (a) and a corresponding broadcast model (b), where nodes 1 and 4 exchange their information through nodes 2 and 3.

To limit the decoding delay, NC schemes often use *generations* [6]. A coding scheme with generation size g enables each packet to be encoded with a maximum of g messages. Increasing the generation size yields more coding opportunities, but it incurs several costs:

- Complex decoding: the number of operations required for decoding of the order of $O(g^2 \log g)$.
- Decoding delay: maximum decoding delay of the order of $O(g + D)$.

[1] Or, alternatively, the "burst" mode in the algebraic approach [2, Sec. VI]. Note that the "continuous" mode in [2, Sec. VI] assumes all links have the same delay, which is not the case herein.

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- Packet overhead: overhead of the order of $O(gD)$.
- Error propagation: an incorrect message may impact g different messages.

Here, we introduce a structured NC scheme with a simple decoding algorithm ($O(1)$), minimum delay ($O(D)$), a small overhead ($O(\log D)$) and low error propagation ($O(1)$).

Next, we define and motivate the network model to be used throughout this work. Following the key concepts in the celebrated [7] and its generalization in [8], we transform a given bidirectional graph into a corresponding directed graph which represents the broadcast model (Fig. 1b). This construction bounds the total amount of information that a relay node can transmit to all adjacent nodes at a given unit time. The motivation of this model is to adhere to the broadcast constraint, which is common in wireless models. Clearly, in the model of Fig. 1b, using a simple store-and-forward scheme yields a rate of $R_1 + R_4 \leq C$, where R_1 and R_4 are the rates of source nodes 1 and 4, respectively, and C is the capacity of each link, while using NC yields the maximum transmission rate of $R_i \leq C$, $i \in \{1, 4\}$. Our objective is thus to derive coding schemes for the broadcast model with bidirectional traffic that achieve the capacity region with minimum delay, and under a variety of topologies and demand structures.

The model suggested in this paper may arise in several networking scenarios. For example, consider a sensor network of simple, low-cost sensors distributed throughout a large area. Since sensors are light and simple, it is reasonable to assume only a single MOD-COD (modulation and coding) regime is used; hence, if sensors can communicate, they can all communicate at a fixed equal rate. Moreover, geographical constraints dictate that either sensors are close enough to communicate, in which case they have bidirectional communication, or they are too far or obscured and cannot communicate. Finally, the wireless medium dictates that a node can broadcast a single message at a single time slot, and this message is heard by all nodes within the reception range of that node.

Fortunately, the broadcast nature of wireless networks is not only a constraint. It can be used to exploit NC opportunities. For example, by exploiting the broadcast ability of the wireless medium, when a coded packet is broadcasted to many adjacent nodes they can all gain new information from a single transmission. For instance, consider an implementation of a video conferencing application in an ad-hoc network over IEEE 802.11, using User Datagram Protocol (UDP). On the one hand, to gain the rate benefits, utilize the broadcast nature of the medium and reduce the delay, an NC scheme is used. On the other hand, implementing NC in such a network introduces several challenges, e.g., coding of unordered messages, delays due handling interferences and errors at the lower layers and packet losses. Namely, such a network suffers from arbitrary random transmission delays, with which the NC scheme must cope. In this work, we offer a coding scheme for the broadcast model, which is applicable to such networks and handles the arbitrary delay efficiently.

Note, however, that unlike much published research on traditional wireless models, we investigate these networks from a higher, network-layer point of view. I.e., we assume the existence of physical layer and data link protocols which handle channel errors and medium access control. Hence, there are no erroneous packets or interferences between links in the models we investigate. These protocols, however, may still result in dropped or lost packets, out of order delivery and arbitrary delays. This work is focused on simple and efficient coding schemes under these circumstances. Thus, our work is applicable to any network were layer 1 and layer 2 protocols ensure a basic (maybe unreliable) packet switching mechanism, and a node's transmission may be overheard by all its neighbors.

A key network characteristic that significantly impacts our ability to give tight results and optimal NC schemes is the demand structure. For example, in a *multicast* scenario [9], [10], all terminal nodes wish to decode all sources. For this case, practical and rate-optimal solutions exist under several network models. However, in a *multiple unicast* scenario, where independent source-destination pairs wish to communicate, the problem is much less tractable, and the general case is still unsolved. In fact, the authors of [11] showed that any acyclic directed network (with a general demand structure) has an equivalent multiple unicast network, stimulating significant interest in the study of such networks. Specifically, unlike the multicast case, linear NC [12] fails to achieve the capacity region in this setting [13]. A few special cases are the capacity region when only XOR operations are permitted [14], and a coding scheme for three unicast sessions that achieves a rate of half the minimum cut using the interference alignment approach [15]. The case of three unicast sessions was also studied in [16], where lower bounds on the connectivity of the network, which allow unity rate, were introduced. Finally, in [17], the authors showed that the existence of a store-and-forward scheme is equivalent to finding edge-disjoint paths. Here, we expand on that result beyond two simple unicast sessions to three bidirectional unicast sessions. Thus, we continue to explore the case of three unicast sessions, but, in our case, the three sessions are bidirectional and experience arbitrary delays.

A network with arbitrary transmission delays was also studied in [18], where an opportunistic NC scheme was examined. An opportunistic NC scheme is a coding scheme that exploits coding opportunities whenever it can [19]. Here, we use a different approach of a structural NC scheme that exploits coding at each transmission without appending the global encoding vector, which poses decoding difficulties in coping with arbitrary transmission delays. The effects of arbitrary transmission delays on NC schemes was also studied in [20], where an NC for Transmission Control Protocol (TCP) sessions in wireless networks was presented.

Additional Related Works

A paramount contribution in this context is COPE [3]. However, there are important differences between the schemes we suggest herein and COPE. First, COPE is an opportunistic scheme, which requires reception reports from the nodes and encodes based on the reports, guessing and searching for coding opportunities. Furthermore, it does not send packets which cannot be decoded at the nodes, loosing some coding

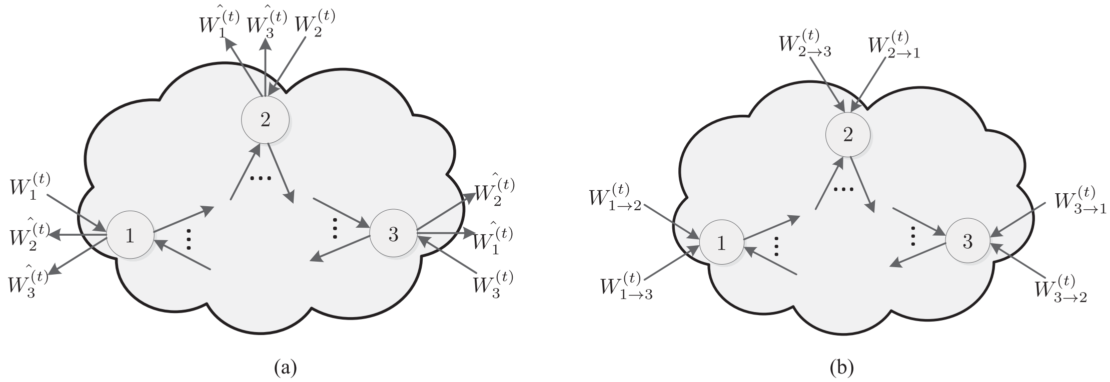

Fig. 2. Schematic illustration of the multicast model (a) and the multiple unicast model (b), where $W_i^{(t)}$ is a message that node i generates at time instant t and that is destined for all the other source nodes, and $W_{i\to j}^{(t)}$ is a message that node i generates at time instant t and that is destined only for node j.

possibilities. Our schemes are structures, do not need searching or reports, and look ahead by sending consecutive coded packets. Second, we define the rate regions for several interesting scenarios, and show that they are achieved by our coding schemes. Finally, COPE does not have the delay guarantees available for our coding schemes.

Several works extended [3] to handle *hard deadline constraints*. For example, [21] suggested discarding expired packets, and prioritizing older packets when coding in order to meet the constraints. Reference [22] suggested sending *uncoded packets* when a receiver has too many packets coded together, and waiting for additional coded packets until all can be decoded together might increase the delay significantly. Reference [23] showed that the problem of finding the best coding vectors while maintaining the deadline constraints is an NP-complete problem, and suggested heuristics to solve it. Thus, this line of works focused on pre-defined, arbitrary deadlines, allowing packets to be lost completely in order to meet them, and suggesting heuristic coding schemes for the problem. Herein, however, we focus on minimizing delay, without packet drops, and give optimal coding schemes under several topologies.

There is a significant body of work on network coding for video applications. Reference [24] suggested network coding for emergency related video streaming. While the scheme improves on uncoded schemes, it does not have the delay or capacity guarantees we have in this paper, and, in general, can have a larger delay as nodes request for packets opportunistically. Reference [25] also suggested opportunistic coding, though the scheme therein focuses on *video aware* coding, that is, adding video quality and specific packet deadlines into the coding choice. [26] focused on maximizing the numbers of layers achieved by each receiver, again, to improve video quality. Reference [27] showed that indeed network coding can improve quality by *limiting the need for synchronization* between senders of a video stream. Our finding in this paper, of defeating arbitrary delay using coding compliments [27] in this sense, with coding and analysis for a wide variety of topologies. Reference [28] studied delay-optimized network coding. However, therein, the focus is on slightly *delaying packet transmission* in order to achieve better coding opportunities and improve video quality. In our scheme, packets are not delayed at all, achieving lowest possible delay, yet, for interesting scenarios, we show that this scheme actually achieves the optimal throughput. Finally, in [29], the authors report on a large scale deployment of a network-coded video streaming application. While the paper reports very good performance, it specifically mentions drawbacks that the coding community is facing: the large overhead involved in implementing *random* network coding in these systems. As mentioned before, the scheme suggested in this paper is deterministic and is specifically targeted at the delay and overhead issues of random network coding.

On top of that, there is a whole line of research on *instantly decodable* network codes [30]–[32]. In these problems, the goal is to construct codes in which every received packet can be decoded instantly, without the need to wait for additional packets. Clearly, in terms of coding delay, such codes minimize the delay and video is one of the main applications. However, it is important to note that this restriction has a hit on capacity, and for example, it is not possible to achieve capacity in star, line-star and general networks with it. Our coding schemes, on the contrary, achieve capacity in star, line-star and some enhancements, with a coding delay of only 1 packet.

Main Contributions:

We study two demand structures, described above, under the suggested model with arbitrary transmission delays. Specifically, we consider bidirectional multicast and multiple unicast [33]–[35] transmissions, both for up to three users. In the first model (Fig. 2a), three users, $1, 2$ and 3, exchange messages in a bidirectional multicast manner through a bidirectional network. Multicast transmission, in which a message is sent to a set of receivers, is a widely used networking technique; one example of this is a video conference between three users. In the second model, each user generates two different messages, one for each of the remaining users, i.e., it exchanges independent messages with two different users in a bidirectional manner through a network (Fig. 2b). For an example of the multiple unicast case, consider a messaging application with several users, where each user communicates with its partners, but each message is addressed to a specific partner. The two paradigms, multicast and multiple unicast, are used as sub-topologies in many different communication networks, from wired, through optical fibers, to wireless ad-hoc networks.

For both demand structures, we achieve the equal rate capacity in the corresponding broadcast models, where each two-way communication is carried out at the same rate. Additionally, our coding schemes are shown to be Real Time (RT) NC schemes, which we define as an NC scheme that facilitates decoding with minimum delay. Such a scheme is suitable for several applications, such as video conferencing and instant messaging.

In the multicast scenario, the benefits of the coding schemes are their practicality and applicability to networks with arbitrary delays. In the multiple unicast scenario, which is generally open, we also extend the current state of the art regarding when NC is optimal and what are the achievable rates.

Our coding schemes are based on a modular approach. We begin by providing simple constructions for line and star topologies (e.g., [8], [36]), after which we use graph-theoretic tools to show how to decompose general networks into the above building blocks. Specifically, constructive algorithms are given to show that bidirectional networks with arbitrary transmission delays can be decomposed into line and star topologies as building blocks, can achieve the maximum rate and can deploy an RT coding scheme with a small overhead, a simple decoding algorithm and low error propagation.

The rest of the paper is organized as follows. In Section II, we present the network model. In Section III, we outline the preliminaries of the coding schemes for the line, star and line-star topologies and then use these coding schemes in Sections IV and V as building blocks to derive a coding scheme for a general bidirectional network with multicast and multiple unicast sessions, respectively. Simulation results are given in Section VI and then some extensions are presented in Section VII. Finally, in Section VIII we summarize the paper with our conclusions.

II. Notation and Problem Setup

A *bidirectional network with delays* is defined as a directed graph, $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\ldots,M\}$ is a set of nodes and $\mathcal{E}\subseteq[1,\ldots,M]\times[1,\ldots,M]$ is a set of bidirectional edges. Each edge $(i,j)\in\mathcal{E}$ represents a directed link from node i to node j with a capacity of C bits per time unit. Since the edges are bidirectional, each edge $(i,j)\in\mathcal{E}$ induces a corresponding edge $(j,i)\in\mathcal{E}$ with the same capacity. Additionally, we consider a set of source nodes $\mathcal{S}\subseteq\mathcal{V}$.

Next, we present a model that allows us to explore the *broadcast* ability of the wireless medium. Therefore, we introduce an equivalent directed graph, $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$, with the same set of source nodes $\mathcal{S}\subseteq\mathcal{V}'$, by splitting each relay node $i\in\mathcal{V}\setminus\mathcal{S}$ into two nodes $\{i,i'\}\subset\mathcal{V}'$ (Fig. 3). Each pair of directed edges (i,j) and (j,i) in $\mathcal{E}$ corresponds to a pair of new directed edges, one entering i, $(j',i)\in\mathcal{E}'$, with capacity C and another leaving i', $(i',j)\in\mathcal{E}'$, with the same

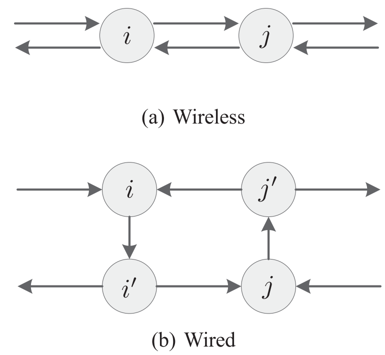

Fig. 3. Diagram showing the conversion of a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ in (a) into a graph $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ in (b), $i,j\in\mathcal{V}\setminus\mathcal{S}$.

capacity. In addition, there is an edge directed from i to i' with capacity C. This edge means every relay-output node i' has only one incoming edge, hence at a given time slot the rank of its input is at most one. If all memory and computation is in i, this means *all outgoing edges* from i' send the same data at a given time slot. This is the broadcast constraint. It is similar to the one used in [7]. Note that a simpler approach would be to define outgoing *hyperedges*, yet we prefer to correct notation which is consistent with [7].

We now define the messages the sources have. In general, there are two types of messages. Multicast messages, intended from a node in $\mathcal{S}$ to all other nodes in $\mathcal{S}$, and unicast messages, intended from a node in $\mathcal{S}$ to a single other node in $\mathcal{S}$. The messages and rates are defined below.

- $W_i^{(t)}$ denotes the multicast message of source i at time t. We assume each message is distributed uniformly over $\{1,\ldots,2^{R_i}\}$. Therefore, R_i denotes the number of bits in one such single packet. Since such packet is sent every unit time, it is also the rate in which multicast messages are generated by node i.
- $W_{i\to j}^{(t)}$ denotes the unicast message of source i intended for node j at time t. Again, it is distributed uniformly over $\{1,\ldots,2^{R_{i\to j}}\}$. Therefore, $R_{i\to j}$ represent the rate in which source i generates unicast messages that are intended to node j.
- $\mathcal{W}_i^t=\{W_i^{(0)},W_i^{(1)},\ldots,W_i^{(t)}\}$ represents the set of multicast messages that was produced by source i up to time t. Similarly, $\mathcal{W}_{i\to j}^t=\{W_{i\to j}^{(0)},W_{i\to j}^{(1)},\ldots,W_{i\to j}^{(t)}\}$ for unicast.
- $X_i^{(t)}$ represents the broadcast binary vector transmitted on all the edges leaving node i $\{(i,j):j=1,\ldots,M,(i,j)\in\mathcal{E}'\}$ at discrete time t.

We assume messages at negative times equal zero, i.e., $W_{i\to j}^{(t)}=0$ and $W_i^{(t)}=0$, $\forall t<0$ and $\forall i,j\in\mathcal{S}$.

Our model consists of arbitrary transmission delays, i.e., $X_i^{(t)}$ is sent from node i to node j in time slot t and yet is received by node j after an arbitrary integer delay, $d_{i,j}^{(t)}$, $(i,j)\in\mathcal{E}'$. The delay is assumed to be bounded by D, i.e., $d_{i,j}^{(t)}\leq D$, $\forall(i,j)\in\mathcal{E}'$ and $\forall t$. We assume there is no delay in the node processors, i.e., the transmission over the edge $(i,i')\in\mathcal{E}'$ has no delay for all $i\in\mathcal{V}'\setminus\mathcal{S}$. Note that the model allows for different delays for different recipients of the same transmission. This is important to capture various error and erasure correction mechanisms at the nodes, which result in different delays. Note further that in most of this paper we focus on the arbitrary delay ensued in such networks, and do not handle packets losses directly. This can be done using a simple extension to [7] and is discussed in Section III-D.

The outgoing transmission from every node at any particular time instant t is a function of the incoming transmissions to that node at earlier time instants and of its own messages. Throughout the paper, we use the operators floor $\lfloor \cdot \rfloor$ and ceiling $\lceil \cdot \rceil$.

We denote by $\mathcal{C}_{i;j}$ the value of the minimum cut between nodes i and j in $\mathcal{G}'$. Due to the bidirectional links, $\mathcal{C}_{i;j} = \mathcal{C}_{j;i}$. Similarly, we denote by $\mathcal{C}_{i;j,l}$ the minimum cut between nodes i and j, l. We denote by $\mathcal{P}_{i;j}$ the set of disjoint paths from node i to node j in $\mathcal{G}'$, where it follows that $|\mathcal{P}_{i;j}| = \frac{\mathcal{C}_{i;j}}{C}$. Further, define

$$h = \frac{\min_{i \in \mathcal{S}} \mathcal{C}_{i;\mathcal{S}\setminus\{i\}}}{C}. \tag{1}$$

Finally, we define a maximum distance in a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ as $L = \max |\mathcal{P}_{i;j}|$, where the maximum is taken with respect to all $\mathcal{P}_{i;j}$, $i, j \in \mathcal{S}$. The maximum distance excludes the edges $(i, i') \in \mathcal{E}'$, $\forall i \in \mathcal{V}' \setminus \mathcal{S}$, since they have no delay. The diameter of the graph is the maximum distance in a network with $\mathcal{S} = \mathcal{V}$. We will use the following definitions throughout the paper.

Definition 1 (Achievable Rate): A rate tuple

$$\left(R_i, \left\{R_{i\to j}\right\}_{j\neq i}\right)_{i=1}^{|\mathcal{S}|} \tag{2}$$

is said to be achievable if every node $j \in \mathcal{S}$ is able to decode messages that are destined for it, i.e., $W_i^{(t)}$ and $W_{i\to j}^{(t)}$ for all $i \in \mathcal{S} \setminus \{j\}$, at rates R_i and $R_{i\to j}$, respectively.

Definition 2 (Equal Rate Capacity Region): The capacity region under the equal rate constraint is defined as the closure of the set of all achievable rate tuples (2), $j \in \mathcal{S}$, with the demands $R_{i\to j} = R_{j\to i}$ and $R_i = R_j$.

Note that the rates in Definitions 1 and 2 refer to an *average rate*, in bits per unit time, regardless of any possible delays or initializations. This is, in other words, the best possible throughput and may be achieved only by averaging over long periods of time. Time sensitive applications, however, require some bounded delay, and cannot tolerate coding schemes which use large generations or block sizes (hence long delays) to maximize throughput. The following definition addresses this issue, rigorously defining a *real time* coding scheme as a one whose delay is bounded by the minimal possible delay (inherent in the topology), plus some independent constant. Interestingly, in the sequel we show real time coding schemes which achieve the equal rate capacity region in some interesting demand structures.

Definition 3 (Real Time (RT)): A coding scheme is said to be RT for a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ with a maximum distance L and an arbitrary delay bounded by D if for all t, *every node $j \in \mathcal{S}$ decodes all messages up to time t, $\left\{\mathcal{W}_i^t, \mathcal{W}_{i\to j}^t\right\}_{i\neq j}$, with a worst-case minimum delay of $LD + t + c$, where c represent the initialization time which is independent of L and D.*

Note that the worst-case delay *of any scheme* on a network with a maximum distance L is bounded by LD. Thus, an RT coding scheme achieves the lower bound up to a constant independent of the network size and the transmission delay. This means that after an initial period of *at most* $LD + c$, an RT coding scheme decodes a new message in each time slot. There is no additional delay due to coding in blocks, generations, or due to nodes waiting for messages from all incoming links. Such coding schemes may continue to hinder performance even after the initialization time. For example, a straight forward implementation of RLNC which waits for all packets produced at time t to arrive before coding (otherwise, the rate is below capacity) will incur a delay of LDt. Trying to mitigate the problem, using generations of length G, will result in a delay of about $LD+t+G$, yet G is not a negligible constant. In fact, it should be relatively high for a high coding gain [6] and may grow with the network size. Hence, the constant c can be referred to as a *coding delay*. The lower c is, the better the coding scheme in that sense. In some schemes, it may scale up with the network size, and hence can be quite large. In an RT scheme, however, it is independent of the network and reflects only constant initialization time. In the schemes we suggest herein, c is at most 1. Moreover, in general the delay in the above mentioned schemes is affected by the longest path in the network. In the schemes suggested herein, the delay to a destination node is affected only by the lengths of the paths to that node.

Finally, the goal is to find an RT coding scheme for a graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ that *achieves the capacity region under an equal rate* assumption. We also require a low overhead. The coding scheme presented here requires only an overhead (as a header) of the order of $O(\log_2 h)$ bits to separate the network into the building blocks, while a RLNC scheme requires $O(h)$ bits to transmit the global encoding vector, where h is the minimum between all the minimum cuts that separate one source from the network.

III. Preliminaries

In this section, we describe the key concepts of the coding scheme for the line, star and line-star topologies. We later use these schemes as building blocks for more complex networks.

A. Line Topology

A line topology of M nodes is defined as a network $\mathcal{G}(\mathcal{V}, \mathcal{E})$ of two source nodes $\mathcal{S} = \{1, M\}$ that exchange messages $\mathcal{W}_1^t$ and $\mathcal{W}_M^t$ through a line of nodes. The order of the nodes is ascending, i.e., relay node $i \in \mathcal{V} \setminus \mathcal{S}$ has two adjacent nodes, node $i-1$ which is nearer to node 1 and node $i+1$ which is nearer to node M. Here, we first transform this network into a corresponding broadcast model (Fig. 4). The coding scheme for this topology was first derived in [7]. For completeness, and since we use it extensively later, we now present a sketch of the scheme. The main result for a line topology is summarized in the following theorem.

Theorem 1: For any line topology with an arbitrary integer delay bounded by D, there exists an RT coding scheme that achieves the equal rate capacity, which is C. Furthermore, the coding scheme has a decoding delay of at most LD and it

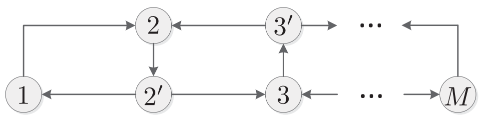

Fig. 4. Graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ of a line topology, where $\mathcal{S} = \{1, M\}$.

Fig. 5. Packet header of the coding scheme for the line topology.

includes a fixed header per transmission of $2\lceil\log_2 2D\rceil$ *bits, independent of* C.

Proof: The minimum cut between the two source nodes is $\mathcal{C}_{1;M} = C$; therefore, we obtain that the equal rate upper bound is $R_i \leq C$, $i \in \mathcal{S}$. Next, we present the coding scheme that achieves this upper bound. Although a field $\mathbb{F}_2$ could be applicable here for the line topology coding scheme, we prefer to use $\mathbb{F}_{2^C}$ throughout since we use it later as a building block together with the star topology coding scheme which requires the field $\mathbb{F}_{2^C}$.

Coding Scheme: The fundamental concept of this coding scheme is that each relay node decodes all the messages.

Source Encoder: source nodes, 1 and M, generate the following transmissions at time instant t

$$X_1^{(t)} = W_1^{(t)}, X_M^{(t)} = W_M^{(t)}. \quad (3)$$

Relay Encoder: each relay node, $r \in \mathcal{V}' \setminus \mathcal{S}$, generates a linear combination in $\mathbb{F}_{2^C}$ at time instant t, as follows

$$X_r^{(t)} = W_1^{(p)} + W_M^{(q)}, \quad (4)$$

where p and q are the indices of the last decoded messages by node r. In the case where only one message was decoded at time instant t, node r simply performs store-and-forward of that message. Each transmission is appended with two indices, p and q, each of length $\lceil\log_2 2D\rceil$ (Fig. 5). This header represents the messages from the sets $\mathcal{W}_1^t$ and $\mathcal{W}_M^t$ that were encoded.

Relay Decoder: to describe the decoding process, we first introduce the following claim. Node r has a subset of the messages $\mathcal{W}_1^t$ that node $r-1$ already decoded at any given time. This is because node $r-1$ only encodes messages it successfully decoded and node r can only receive messages generated by node 1 from node $r-1$, due to the topology of the network. Similarly, node r has a subset of the messages $\mathcal{W}_M^t$ that node $r+1$ already decoded. This means that all transmissions originating from source M which are known to node $r-1$ are surely known to node r.

Therefore, to decode a new message, $W_1^{(p)}$, from the incoming transmission $X_{r-1}^{(t)}$, node r subtracts the message $W_M^{(q)}$ which it already decoded, as follows

$$X_{r-1}^{(t)} - W_M^{(q)} = W_1^{(p)}. \quad (5)$$

Similarly, for a transmission from node $r+1$,

$$X_{r+1}^{(t)} - W_1^{(p)} = W_M^{(q)}. \quad (6)$$

Sink/Source Decoder: for each time instant t, decoder 1 subtracts the message $W_1^{(p)}$, which it generated before time instant t, to decode the information $W_M^{(q)}$. Similarly, node M subtracts $W_M^{(q)}$ to decode $W_1^{(p)}$.

Analysis: Source node $i \in \mathcal{S}$ transmits a new message $W_i^{(t)}$ which consists of C bits at each time instant t. The sink of this message, node $j \in \mathcal{S} \setminus \{i\}$, decodes it after a maximum delay of LD, where $L = M - 1$. Therefore, this coding scheme achieves the capacity rate for this network, which is C. A header of $\lceil\log_2 2D\rceil$ bits is sufficient to identify each message $W_1^{(t)}$ for decoding it at the next hop, since it can encode $2D$ different messages. Moreover, after an initialization time of LD time units a new message will be decoded at each time instant. This is because a message that was transmitted at time instant t has a worst-case decoding delay of $LD+t$. Therefore, for all t, messages $\mathcal{W}_i^t$ are decoded by node $j \in \mathcal{S} \setminus \{i\}$ at time instant $LD + t$. As a result, this coding scheme is an RT coding scheme. Note that under this scheme perfect pipelining is maintained; after an initial delay, a new message is decoded at each time instant. ■

TABLE I

TRANSMISSION ANALYSIS OF THE LINE TOPOLOGY CODING SCHEME FOR $M = 4$ AND $D = 1$

Time slot	Node 1	Node 2	Node 3	Node 4
t=0	$W_1^{(0)}$	-	-	$W_4^{(0)}$
t=1	$W_1^{(1)}$	$W_1^{(0)}$	$W_4^{(0)}$	$W_4^{(1)}$
t=2	$W_1^{(2)}$	$W_1^{(1)} + W_4^{(0)}$	$W_4^{(1)} + W_1^{(0)}$	$W_4^{(2)}$

The main concept of this coding scheme is that every relay node decodes all the messages. Generally, this is not a standard NC scheme requirement. However, in our model of a network with arbitrary transmission delays it is imperative to avoid error propagation and to enable fast decoding. For example, consider the network depicted in Fig. 1b of a line topology with $M = 4$. Node 1 transmits $W_1^{(t)}$ at time instant t. That message, $W_1^{(t)}$, is first decoded by node 2, then by node 3 and, finally, by node 4. Each node that receives an encoded transmission is able to decode $W_1^{(t)}$ since it was encoded with a message it already knows. Table I describes the transmission of each node for $D = 1$, i.e., a network with unit delay.

In this model, i.e., a line network $\mathcal{G}(\mathcal{V}, \mathcal{E})$, applying our coding scheme requires $|\mathcal{V}| - 2$ transmissions by the relay nodes for each pair of messages, $W_1^{(t)}, W_M^{(t)}$, where using a simple store-and-forward scheme requires $2(|\mathcal{V}| - 2)$ transmissions. Note that a dropped message does not effect the performance of this coding scheme since a message is encoded only if it was first successfully decoded, i.e., there is no error propagation. Additionally, this coding scheme has a simple decoding algorithm, which requires only one operation, and a small header of $\lceil\log_2 2D\rceil$ bits.

B. Star Topology

A star topology of three source nodes is defined by a network $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{S} = \{1, 2, 3\}$ and all the source nodes in $\mathcal{S}$ try to communicate through a single relay node, 4.

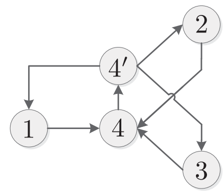

Fig. 6. Graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ of a star topology, where $\mathcal{S} = \{1, 2, 3\}$.

Here, we first transform this network into a corresponding broadcast model (Fig. 6). Our main result for the star topology is summarized in the following theorem.

Theorem 2: For a star topology with an arbitrary integer delay bounded by D, there exists an RT coding scheme that achieves the equal rate capacity, which is $\frac{C}{2}$. Furthermore, the coding scheme includes a fixed header per transmission of $3\lceil \log_2 2D \rceil + 1$ bits.

Proof: The minimum cut between source node 1 and source nodes 2, 3 is $\mathcal{C}_{1;2,3} = C$. Therefore, we obtain that $R_2 + R_3 \leq C$ and the equal rate upper bound is $R_i \leq \frac{C}{2}$, $\forall i \in \mathcal{S}$. Next, we present the coding scheme that achieves this upper bound.

Coding Scheme: Similar to the line topology coding scheme in Section III-A, the relay node decodes all the messages. We choose a priori two non-zero triplets $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$ of coefficients over the field $\mathbb{F}_{2^C}$ that satisfy

$$\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \neq 0, \quad \forall i, j \in \{1, 2, 3\}, i \neq j. \tag{7}$$

Source encoder: source nodes, 1, 2 and 3, generate the following transmissions at time instant t

$$X_1^{(t)} = W_1^{(t)}, \quad X_2^{(t)} = W_2^{(t)}, \ X_3^{(t)} = W_3^{(t)}. \tag{8}$$

Relay Encoder: relay node 4 generates two different transmissions (one with coefficients a and the other with coefficients b),

$$X_4^{(t)} = a_1 W_1^{(p)} + a_2 W_2^{(q)} + a_3 W_3^{(u)}, \tag{9}$$
$$X_4^{(t+1)} = b_1 W_1^{(p)} + b_2 W_2^{(q)} + b_3 W_3^{(u)}, \tag{10}$$

where p, q and u are the indices of the *last decoded messages* by node 4. Note that from the definition of the source encoder above, the relay receives uncoded messages and hence can always create linear combinations (9) and (10). If, due to delay, fewer messages are received at the relay, the linear combinations include the recently received ones, be it only two or just one, in which case the result is an uncoded packet. The three indices p, q and u, each of length $\lceil \log_2 2D \rceil$, are appended as metadata to each transmission. Additionally, we append a bit k that indicates which set of coefficients were encoded, a or b (Fig. 7). This header represents the messages from the sets $\mathcal{W}_1^t, \mathcal{W}_2^t$ and $\mathcal{W}_3^t$ that were encoded, as well as the set of coefficients to allow decoding at the next hop.

Relay Decoder: relay node 4 receives non-coded transmissions and it is able to decode the messages $W_1^{(t)}, W_2^{(t)}$ and $W_3^{(t)}$ that it receives.

Sink/Source Decoder: to describe the decoding process, we first introduce the following claim. Node 4 has a subset of

p	q	u	k	$Data$

Fig. 7. Packet header of the coding scheme for the star topology.

the messages $\mathcal{W}_1^t$ that node 1 has at any given time. This is because node 1 generates messages $\mathcal{W}_1^t$. Similarly, node 4 has a subset of the messages $\mathcal{W}_2^t$ and $\mathcal{W}_3^t$ that nodes 2 and 3 have, respectively.

Therefore, since the coefficients, a and b, were chosen according to (7), source node 1 is able to decode two new messages, $W_2^{(q)}$ and $W_3^{(u)}$, from the incoming transmissions, $X_4^{(t)}$ and $X_4^{(t+1)}$, by solving a system of independent linear equations over the field $\mathbb{F}_{2^C}$. Similarly, node 2 is able to decode messages $W_1^{(p)}$ and $W_3^{(u)}$ and node 3 is able to decode $W_1^{(p)}$ and $W_2^{(q)}$.

Analysis: For each new message $W_1^{(t)}$ from the field $\mathbb{F}_{2^C}$ generated by source node 1 at time instant t, relay node 4 transmits two encoded transmissions. The message, $W_1^{(t)}$, is then decoded by the sink nodes 2 and 3 after a maximum decoding delay of LD, where $L = 2$. As such, we set the rate of each source node $i \in \mathcal{S}$ to $R_i = \frac{C}{2}$. Therefore, this coding scheme achieves the equal rate capacity for this network. Furthermore, after an initialization time of $LD + 1$ time units a new message will be decoded at each time instant. This is because, a message that was transmitted at time instant t has a worst-case decoding delay of $LD + t + 1$. Therefore, for all t, messages $\mathcal{W}_i^t$ are decoded by node $j \in \mathcal{S} \setminus \{i\}$ at time instant $LD + t + 1$. As a result, this coding scheme is an RT coding scheme. ■

Similar to the line topology coding scheme from Section III-A, the idea behind this coding scheme is that the relay node decodes all the messages. For example, consider the network depicted in Fig. 6. Nodes 1 and 2 encode $W_1^{(t)}$ and $W_2^{(t)}$ and transmit $X_1^{(t)}$ and $X_2^{(t)}$ at time instant t, respectively. Those messages, $W_1^{(t)}$ and $W_2^{(t)}$, are first decoded by node 4, and then by node 3. Each node receives two independent encoded packets with two unknown messages and, therefore, is able to decode two new messages.

In this model, i.e., a star network $\mathcal{G}(\mathcal{V}, \mathcal{E})$, applying our coding scheme requires two transmissions by the relay node for each triplet of messages, $W_1^{(t)}, W_2^{(t)}, W_3^{(t)}$, where using a simple store-and-forward scheme requires three transmissions. Note that a dropped message does not effect the performance of this coding scheme since a message is encoded only if it was first successfully decoded, i.e., it has no error propagation. Additionally, this coding scheme has a simple decoding algorithm, which requires to inverse a 2×2 matrix, and a small header of $3\lceil \log_2 2D \rceil + 1$ bits.

C. Line-Star Topology

The line-star topology is generally defined as a combination of line and star topologies (Fig. 8), in which a line of nodes connects a source node to the star topology structure. Each line topology from source node i to the star topology structure is numbered in ascending order, i.e., relay node $r \in \mathcal{V} \setminus \mathcal{S}$ has two adjacent nodes, node $r - 1$ which is nearer to node i and

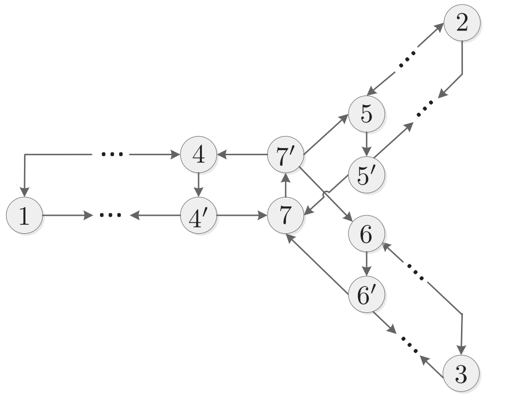

Fig. 8. Graph $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ of a line-star topology, where $\mathcal{S}=\{1,2,3\}$. A line topology is formed between nodes 1 and 4, and a star topology is formed around node 5.

node $r+1$ which is nearer to the star topology structure. This topology is an extension of the star topology to the case where multiple nodes connect each source node to the star topology structure.

Theorem 3: For a line-star topology with an arbitrary integer delay bounded by D, there exists an RT coding scheme that achieves the equal rate capacity, which is $\frac{C}{2}$. Furthermore, the coding scheme includes a fixed header per transmission of $3\lceil\log_2 2D\rceil+1$ *bits.*

Proof: The minimum cut between source node 1 and source nodes 2, 3 is $\mathcal{C}_{1;2,3}=C$. Therefore, we obtain that $R_2+R_3\leq C$ and the equal rate upper bound is $R_i\leq\frac{C}{2}$, $\forall i\in\mathcal{S}$. Next, we present the coding scheme that achieves this upper bound.

Coding Scheme: The encoding process of this coding scheme is similar to that for the star topology coding scheme in Section III-B. However, the relay nodes decoding process is different. Specifically, we derive two different decoding processes for the star topology relay node and the line topology relay nodes. Similar to for the star topology coding scheme, we choose two non-zero triplets $a=[a_1,a_2,a_3]$ and $b=[b_1,b_2,b_3]$ of coefficients over the field $\mathbb{F}_{2^C}$ that satisfy

$$\begin{vmatrix}a_i & a_j\\ b_i & b_j\end{vmatrix}\neq 0,\quad \forall i,j\in\{1,2,3\}, i\neq j. \tag{11}$$

Source Encoder: source nodes 1, 2 and 3 generate the following transmissions at time instant t

$$X_1^{(t)}=W_1^{(t)},\quad X_2^{(t)}=W_2^{(t)},\ X_3^{(t)}=W_3^{(t)}. \tag{12}$$

Relay Encoder: relay node r generates two different transmissions (one with coefficients a and the other with coefficients b),

$$X_r^{(t)}=a_1W_1^{(p)}+a_2W_2^{(q)}+a_3W_3^{(u)}, \tag{13}$$
$$X_r^{(t+1)}=b_1W_1^{(p)}+b_2W_2^{(q)}+b_3W_3^{(u)}, \tag{14}$$

where p,q and u are the indices of the messages last decoded by node r. Again, to see how can the relay create the combinations above, note that it encodes only messages it recently decoded (the three types of decoders required in this scheme are described below, together with an explanation how they are able to decode each packet). The three indices p,q and u, each of length $\lceil\log_2 2D\rceil$, are appended as metadata to each transmission. Additionally, we append a bit k that indicates which set of coefficients were encoded, a or b (Fig. 7). This header represents the messages from the sets $\mathcal{W}_1^t,\mathcal{W}_2^t$ and $\mathcal{W}_3^t$ that were encoded, as well as the set of coefficients.

Star Topology Relay Decoder: the relay node subtracts two messages from the incoming encoded transmission to decode a new message, i.e., the topology yields that each incoming transmission is encoded with two known messages.

Sink/Source Decoder: source node i receives two independent equations and is able to decode two messages $W_j^{(p)},W_l^{(q)}$, $j,l\in\mathcal{S}\setminus\{i\}$, where p and q are the indices of the messages in the coded vector it received.

Line Topology Relay Decoder: relay node r has two different kinds of incoming transmissions. From the incoming transmission from relay node $r-1$ it decodes a new message in the same manner as the star topology relay decoder and from relay node $r+1$ it decodes two new messages from two incoming transmissions, in the same way as the sink/source decoder.

Analysis: For each new message $W_1^{(t)}$ from the field $\mathbb{F}_{2^C}$ generated by source node 1 at time instant t, relay node r transmits two encoded transmissions. The message is then decoded by the sink nodes 2 and 3 after a maximum decoding delay of LD. As such, we set the rate of each source node $i\in\mathcal{S}$ to $R_i=\frac{C}{2}$. Therefore, this coding scheme achieves the equal rate capacity for this network. Furthermore, after an initialization time of $LD+1$ time units, a new message will be decoded at each time instant. This is because a message that was transmitted at time instant t has a worst-case decoding delay of $LD+t+1$. Therefore, for all t, messages $\mathcal{W}_i^t$ are decoded by node $j\in\mathcal{S}\setminus\{i\}$ at time instant $LD+t+1$. As a result, this coding scheme is an RT coding scheme. ■

In this model, i.e., a line-star network $\mathcal{G}(\mathcal{V},\mathcal{E})$, applying our coding scheme requires two transmissions by each relay node for each triplet of messages, $W_1^{(t)},W_2^{(t)},W_3^{(t)}$, where using a simple store-and-forward scheme requires three transmissions. Note that a dropped message does not effect the performance of this coding scheme since a message is encoded only if it was first successfully decoded, i.e., there is no error. Additionally, this coding scheme has a simple decoding algorithm and a small header.

The topologies of line, star, and line-star are all from the same family in the sense that they all represent networks with a minimum cut of one between each two source nodes. Additionally, by applying a simple store-and-forward scheme to the original model, i.e., the network $\mathcal{G}(\mathcal{V},\mathcal{E})$, of each of the topologies, we can achieve the maximum rate. However, by using NC, we achieve the minimum number of transmissions, i.e., efficient energy expenditure with maximum rate. The coding schemes presented in Sections III-A, III-B and III-C hold the properties of RT and are *innovative* [37]. A coding scheme is called innovative if each incoming transmission to source node $i\in\mathcal{S}$ is not contained in the span of messages previously received by i. Furthermore, the line topology coding scheme in Section III-A also holds the property of *instantly decodable*, meaning that it is a coding scheme in which a new message is decoded for each incoming transmission, i.e., an RT coding scheme with $c=0$ (without initialization time). In the rest of the paper, we use these coding schemes as building blocks and exploit their properties to construct a coding scheme for a general network.

D. Packet Losses

In [7, Ch. 3], the authors show how the line topology coding scheme can handle packet losses. Specifically, they suggest to add a simple mechanism of acknowledgment to each message to enable a retransmission policy which handles packet losses. This mechanism includes two indices which are appended to each transmitted message to indicate which messages were already decoded.

For example, if node i already decoded messages $W_1^{(0)},\ldots,W_1^{(3)},W_M^{(0)},\ldots,W_M^{(5)}$ it appends indices 3 and 5, which indicate to nodes $i-1$ and $i+1$, the recipients of the message, that all the messages generated by node 1 up to index 3 and all the messages generated by node M up to index 5 are already decoded. Subsequently, each node choses messages that are required by a neighbor node and stop transmitting unnecessary messages, e.g., node $i-1$ stops encoding messages $W_1^{(0)},\ldots,W_1^{(3)}$ in the messages it transmits and node $i+1$ stops encoding messages $W_M^{(0)},\ldots,W_M^{(5)}$.

In this paper, we argue that the star and line-star topologies can be extended in a similar fashion, hence the solutions we suggest can handle packet losses in the same way. This extension includes appending each transmitted message with *three* indices. Each index is used to indicate which messages were already decoded by that node. For example, if node i already decoded messages $W_1^{(0)},\ldots,W_1^{(3)},W_2^{(0)},\ldots,W_2^{(5)},W_3^{(0)},\ldots,W_3^{(7)}$ it append indices $3,5$ and 7. Those indices are used by the neighboring nodes to determine which messages are required by node i. Specifically, they enable the neighboring nodes to retransmit messages that are lost.

This supplement to the coding schemes assures robustness to packet losses and can be used by line, star and line-star topologies. Therefore, it is also applies to any general network that uses those topologies as building blocks.

IV. Multicast Network

In this section, we combine the line topology from Section III-A and the line-star topology from Section III-C and use them as building blocks to present a new coding scheme for a general multicast network.

A general multicast network is defined by a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with a set of source nodes $\mathcal{S}=\{1,2,3\}$ and a corresponding broadcast model, $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ (Fig. 2a). In the following, we show that every network is decomposable into line-star and ring sub-networks, where a ring is defined as the union of three special line topologies. A special line topology is valid for a ring sub-network if deleting the line connecting two sources from the graph would not reduce the minimum cut of the remaining source node, e.g., if deleting $P_{1;2}$ would not reduce $\mathcal{C}_{3;1,2}$. Each network may have many decompositions with such building blocks. However, we prove that there exists at least one decomposition that, using the coding schemes we already derived for the building blocks, achieves the equal rate capacity for a general network, in which three source nodes communicate bidirectionally in a multicast manner.

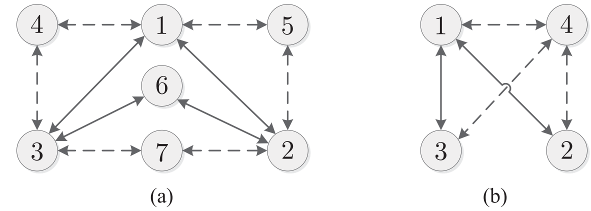

Fig. 9. Schematic illustrations of (a) a network with two elements in $\mathcal{R}$ (two rings, one in the dashed lines) and (b) a network with two elements in $\mathcal{Q}$ (two line-star topologies, one in the dashed lines).

A. Capacity and Coding for a Multicast Network Based on Line and Star Topologies

Our main result is summarized in the following theorem.

Theorem 4: For any multicast network $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ with three source nodes and an arbitrary integer delay bounded by D, there exists an RT coding scheme that achieves the equal rate capacity, which is $\frac{hC}{2}$. Furthermore, the coding scheme includes a fixed header per transmission of $3\lceil\log_2 2D\rceil+1+\lceil\log_2 h\rceil$ bits.

Converse Proof of Theorem 4: The equal rate upper bound is obtained by the standard minimum-cut arguments [38]. We assume, without loss of generality, that $h=\frac{\mathcal{C}_{i;\mathcal{S}\setminus\{i\}}}{C}$ for some $i\in\mathcal{S}$. Therefore, we get an upper bound of $R_j+R_l\leq hC$, $j,l\in\mathcal{S}\setminus\{i\}$, and the equal rate upper bound is $R_i\leq\frac{hC}{2}$. Additionally, since the maximum distance in the graph $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ is L, we derive that the lower bound of the worst-case decoding delay is LD. This means that the maximum rate of any coding scheme is $R_i\leq\frac{hC}{2}$, $\forall i\in\mathcal{S}$, and that any coding scheme with rate $R_i\leq\frac{hC}{2}$ cannot guarantee delay lower than LD. ■

To prove the achievability of Theorem 4, we first introduce Lemma 1, which shows how to partition each network $\mathcal{G}'$ into sub-networks of line and line-star. The line sub-networks are represented by a set of *rings* $\mathcal{R}$, where each $r\in\mathcal{R}$ is a set of edges defined in the following definition.

Definition 4 (Ring): A ring $r\in\mathcal{R}$ is a sub-network in a graph $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ with three source nodes, $\mathcal{S}=\{1,2,3\}$, and a set of edges, $r\subseteq\mathcal{E}'$, which form three bidirectional paths of capacity C between each two source nodes under the condition that deleting a path between i and j from the graph would not reduce the minimum cut of the remaining source node l, i.e., $\mathcal{C}_{l;i,j}$, $i,j,l\in\mathcal{S}$.

Each ring r in a graph contributes a rate of C to each source node by using the line topology coding scheme in Section III-A at each bidirectional path. We therefore wish to identify as many independent rings as possible, namely, $r_1\bigcap r_2=\emptyset$, $\forall r_1,r_2\in\mathcal{R}$. For example, consider the two rings in Fig. 9a.

On the same graph $\mathcal{G}'(\mathcal{V}',\mathcal{E}')$ we also define the line-star sub-network. The line-star sub-network is represented by a set of edges $\mathcal{Q}$, $\mathcal{Q}\bigcap\mathcal{R}=\emptyset$. Each $q\in\mathcal{Q}$ is defined by a union

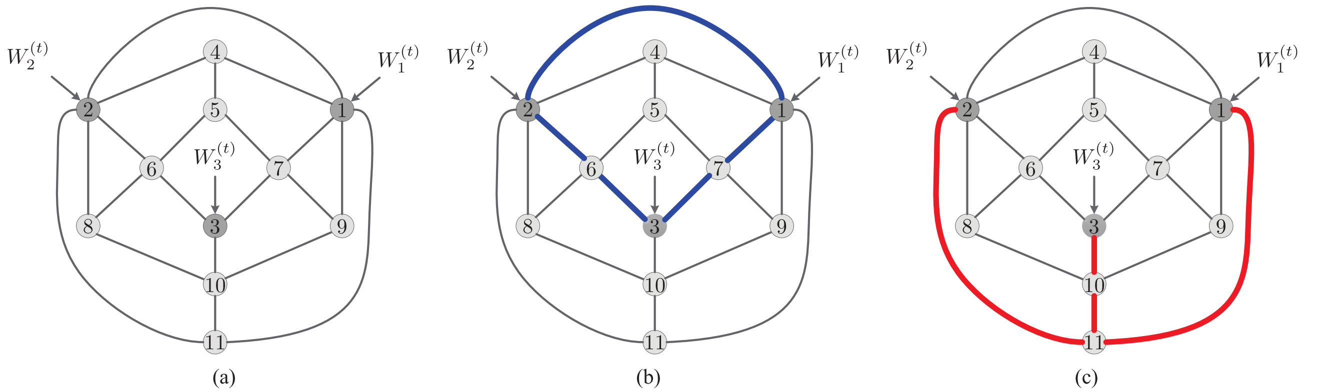

Fig. 10. A network $\mathcal{G}$ is shown in (a), where all the source nodes 1, 2 and 3 communicate in a bidirectional multicast manner. By using the ring and line-star building blocks approach from Section IV, we get the set $\mathcal{R}$, which represents the ring sub-network, illustrated in (b), and the set $\mathcal{Q}$, which represents the line-star sub-networks, as illustrated in (c).

of two bidirectional paths, both leaving the same source node, but each destined to another source node (Fig. 9b).

Note that each ring can also be an element in $\mathcal{Q}$. However, each $q \in \mathcal{Q}$ contributes only a rate of $\frac{C}{2}$ to each source node when using the line-star topology coding scheme from Section III-C. Thus, although there exist many decompositions of $\mathcal{R}$ and $\mathcal{Q}$ in a network $\mathcal{G}'$, in the following lemma we show that by first finding the maximum number of rings, we can assure that there exist enough rings in $\mathcal{R}$ and star-lines in $\mathcal{Q}$ to achieve the equal rate upper bound.

Lemma 1: For a network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, there exist $\mathcal{R}$ and $\mathcal{Q}$ such that $|\mathcal{R}| + \frac{|\mathcal{Q}|}{2} \geq \frac{h}{2}$, where $\mathcal{R} \cap \mathcal{Q} = \emptyset$. Namely, $\mathcal{R}$ and $\mathcal{Q}$ have no mutual edges.

Proof of Lemma 1: The proof is by construction. First, we search for the maximum number of rings, $|\mathcal{R}|$. Then, we construct a network $\mathcal{G}''$ that is the network $\mathcal{G}'$ without $\mathcal{R}$, i.e., we remove the edges in $\mathcal{R}$ from $\mathcal{G}'$. Without loss of generality, we assume that between source nodes 1 and 2 there are no more paths, such that deleting them from $\mathcal{G}''$ would not reduce $\mathcal{C}_{3;1,2}$. We can always find a pair of source nodes that satisfies this condition, because otherwise we could increase $\mathcal{R}$. Finally, we find a set of new paths $\mathcal{Q}$ in $\mathcal{G}''$, where $|\mathcal{Q}| = \frac{\mathcal{C''}_{1;2}}{C}$, where $\mathcal{C''}_{1;2}$ is the minimal cut between nodes 1 and 2 in the graph $\mathcal{G}''$. Each path, $P_{1;2} \in \mathcal{P}_{1;2}$, from node 1 to node 2 shares at least one common node with a special path, $P_{1;3} \in \mathcal{P}_{1;3}$, from node 1 to node 3. This special path, $P_{1;3}$, shares no common nodes with all the other paths, $\mathcal{P}_{1;2} \backslash \{P_{1;2}\}$, between nodes 1 and 2. At least one special path $P_{1;3}$ exists that corresponds to each selection of $P_{1;2}$, since otherwise deleting $P_{1;2}$ would not reduce $\mathcal{C}_{3;1,2}$. As a result, finding the path $P_{1;2}$ and one of the special paths $P_{1;3}$ that corresponds to it is equivalent to finding a line-star sub-network. The line-star sub-network consists of the union between $P_{1;2}$ and $P_{1;3}$, i.e., the union of $P_{1;2}$ and $P_{1;3}$ is a $q \in \mathcal{Q}$. Therefore,

$$|\mathcal{Q}| \overset{(a)}{=} \frac{\mathcal{C''}_{1;2}}{C} \tag{15}$$

$$\overset{(b)}{=} \frac{\min\{\mathcal{C''}_{1;2,3}, \mathcal{C''}_{2;1,3}\}}{C} \tag{16}$$

$$\overset{(c)}{=} \frac{\min\{\mathcal{C}_{1;2,3}, \mathcal{C}_{2;1,3}\}}{C} - 2|\mathcal{R}| \tag{17}$$

$$\overset{(d)}{\geq} h - 2|\mathcal{R}|, \tag{18}$$

where (a) follows from the fact that there are $\frac{\mathcal{C''}_{1;2}}{C}$ paths $P_{1;2}$ that have a corresponding path $P_{1;3}$ that represent a line-star sub-network. (b) is true since node 3 can only be in one of the cuts separating nodes 1 and 2, and $\mathcal{C}_{1;2}$ is the minimum of all the cuts separating them. (c) is the transition to the network $\mathcal{G}'$ and (d) follows from (1). ∎

Next, we provide the achievability proof for Theorem 4.

Achievability Proof of Theorem 4: Using the line topology coding scheme from Section III-A at each path in a ring $r \in \mathcal{R}$ yields a rate of $R_i = |\mathcal{R}|C$, $\forall i \in \mathcal{S}$. Furthermore, using the line-star topology coding scheme in Section III-C at each $q \in \mathcal{Q}$ yields a rate of $R_i = \frac{|\mathcal{Q}|C}{2}$, $\forall i \in \mathcal{S}$. Therefore, using Lemma 1 and the fact that $\mathcal{R} \cap \mathcal{Q} = \emptyset$, we obtain that $R_i \geq \frac{hC}{2}$, and since this is an upper bound, we have an equality. Moreover, this coding scheme has a maximum decoding delay of $LD + t + 1$ for all t, by using a header of $3\lceil \log_2 2D \rceil + 1$ bits, according to the coding scheme of the star topology from Section III-B, and of another $\lceil \log_2 h \rceil$ bits to distinguish between the disjoint line and line-star sub-networks. Therefore, we obtain an RT NC scheme that achieves the equal rate capacity. ∎

B. Example

In this subsection, we show an example of a multicast network, illustrated in Fig. 10. This network is first decomposed into ring and line-star sub-networks, and then we use the coding schemes of those two canonical topologies to obtain a new coding scheme that achieves the equal rate capacity. In Fig. 10b, we visualize the ring sub-networks. In this example, only one element in $\mathcal{R}$ exists, since there is only one path from node 1 to node 3 for which its deletion from the graph would not reduce the minimum cut of node 2, i.e., $\mathcal{C}_{2;1,3}$. Note that this is not the only choice for a ring in the network, e.g., a different choice, $\mathcal{R}'$, includes nodes $1, 2, 3, 6, 7$ and 11 and all the direct links that connect them. As a converse example, consider a path from node 1 to node 3 through nodes 9 and 10. This path cannot be a part of a ring sub-network since deleting it would reduce the minimum cut of node 2, i.e., $\mathcal{C}_{2;1,3}$.

In Fig. 10c, we present the line-star sub-networks. In another case where $\mathcal{R}'$ is chosen, a line-star topology may include nodes $1, 2, 3, 8, 9$ and 10 and all the direct links that connect

Algorithm 1 Find $\mathcal{R}$

```
R ← ∅
while True do
    Solve problem (20) for each pair i, j ∈ S.
    Find I'_k which is the input flow to node k after removing the cyclic flow, k ∈ S.
    if I'_k = 0 for any k ∈ S then
        Break.
    end if
    For each pair i, j ∈ S, choose randomly one path, P_{i;j}, out of f between nodes i and j.
    Add the three paths P_{i;j} for each pair i, j ∈ S to R and remove them from the network.
end while
```

them. By using those sub-networks as building blocks, we achieve a rate of $R_i = 1.5C, \forall i \in \{1,2,3\}$, which is also the equal rate upper bound of this network, i.e., $\frac{h}{2}$, where $h = 3C$.

C. Algorithms

To find the set $\mathcal{R}$, we introduce an optimization problem. We denote by $\mathcal{O}_j$ and $\mathcal{I}_j$ the output and input flows from node j, respectively. Furthermore, the flow, denoted by $\mathbf{f}$, is a binary vector of length $|\mathcal{E}'|$ that represents the flow in each edge, where an element $f_{(i,j)} \in \mathbf{f}$ represents a binary flow in edge $(i,j) \in \mathcal{E}'$.

In this problem, we would like to find the maximum number of disjoint paths from node i to node j whose deletion from the network would not reduce the maximum number of disjoint paths from nodes l to nodes i, j, i.e., $\mathcal{C}_{l;i,j}$, where $i, j, l \in \mathcal{S}$. Since each path is bidirectional, we demand maximal input flow to node l, i.e.,

$$\mathcal{I}_l = \frac{\mathcal{C}_{l;i,j}}{C}. \tag{19}$$

Additionally, we demand that $\mathcal{O}_l = 0$, i.e., that the entire flow would originate at nodes i and j and that $\mathcal{I}_i = 0$ since we are interested in finding a flow that terminates at nodes j and l. With condition (19) satisfied, we maximize the flow that originates at node i and terminates at node j, i.e., the flow that is consumed by node j, $\mathcal{I}_j - \mathcal{O}_j$. Since $\mathbf{f}$ is a binary flow and each relay, $m \in \mathcal{V}' \setminus \mathcal{S}$, has equal input and output flows, we get two sets of disjoint paths. First, paths that terminate at node l and satisfy condition (19), and second, paths from i to j. Because the two sets are disjoint, even if we remove the edges associated with the second set, i.e., paths from i to j, condition (19) will still be satisfied. We thus have the following optimization problem, $\forall i, j, l \in \mathcal{S}$,

The result of applying this optimization problem is a binary flow that satisfies the conditions of the optimization problem. From this flow, we subtract the cyclic flow, which is defined as the flow that originates and terminates at the same node, i.e., paths from node j to node j that do not pass through nodes i and l, $i, j, l \in \mathcal{S}$. After the substraction, we obtain a new input flow to node j, $\mathcal{I}'_j$. This flow, $\mathcal{I}'_j$, represents the maximum number of paths between nodes i and j, where condition (19) is satisfied even if we delete those paths from the network, i.e., we remove their edges from the graph. Therefore, they are applicable to the set $\mathcal{R}$. Algorithm 1 describes how to find all of the paths in $\mathcal{R}$.

Algorithm 2 Find $\mathcal{Q}$

```
Remove the edges associated with R from the graph.
Q ← ∅
while True do
    if |Q| ≥ h − 2|R| then
        Break.
    end if
    Find two source nodes, i, j ∈ S, which have no path between them that deleting it from the graph would not reduce C_{l;i,j}, l ∈ S \ {i, j}.
    Find a path between nodes i and j, P_{i;j}, for which deleting it from the network would minimally reduce C_{l;i,j}.
    Remove the edges associated with P_{i;j} from the graph.
    Find all the paths from node l to node i in the new network and delete them from the network, i.e., remove their edges from the graph.
    Restore the path P_{i;j} to the network and find a path P_{i;l} in the new network.
    Add the union of P_{i;l} and P_{i;j} to Q and remove it from the network, i.e., remove the edges associated with Q from E'.
end while
```

$$\begin{aligned}
&\underset{\mathbf{f}}{\text{maximize}}\ \mathcal{I}_j - \mathcal{O}_j \\
&\text{subject to}\ \mathcal{O}_m = \sum_{n:(m,n)\in\mathcal{E}'} f_{(m,n)}, \quad m = 1\ldots,|\mathcal{V}| \\
&\qquad\qquad \mathcal{I}_m = \sum_{n:(n,m)\in\mathcal{E}'} f_{(n,m)}, \quad m = 1\ldots,|\mathcal{V}| \\
&\qquad\qquad \mathcal{I}_l = \frac{\mathcal{C}_{l;i,j}}{C} \\
&\qquad\qquad \mathcal{I}_i = 0 \\
&\qquad\qquad \mathcal{O}_l = 0 \\
&\qquad\qquad \mathcal{O}_i + \mathcal{O}_j = \mathcal{I}_j + \mathcal{I}_l \\
&\qquad\qquad \mathcal{O}_k = \mathcal{I}_k,\ k = 4,\ldots,|\mathcal{V}|.
\end{aligned} \tag{20}$$

Next, we present Algorithm 2, which describes how to find the line-star sub-networks in the graph, $\mathcal{Q}$. This algorithm is based on the assumption that we already deleted the set $\mathcal{R}$ from the network, i.e., we removed the edges associated with $\mathcal{R}$ from the graph. Therefore, there exists at least one pair of source nodes i and j that have no path between them whose deletion would not reduce $\mathcal{C}_{l;i,j}$. After finding i and j, we search for a path $P_{i;j} \in \mathcal{P}_{i;j}$ whose deletion from the graph would cause the smallest possible decrement to $\mathcal{C}_{l;i,j}$, this is because we would like to avoid crossing other paths. Since deleting $P_{i;j}$ results in a decrement of $\mathcal{C}_{l;i,j}$, there exists at least one special path, $P_{i;l} \in \mathcal{P}_{i;l}$, from node i to node l that does not intersect with any of the other paths, $\mathcal{P}_{i;j} \setminus \{P_{i;j}\}$, from i to j, i.e., it only shares a common node with $P_{i;j}$. To find this special path, we delete all the remaining paths

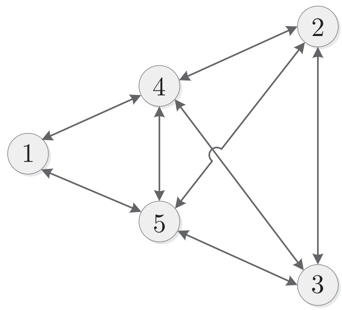

Fig. 11. Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with two star sub-networks, where $\mathcal{S} = \{1, 2, 3\}$.

from node i to node l. By now, we have deleted from the network the path $P_{i;j}$ and all the remaining paths from node i to node l. Next, by restoring $P_{i;j}$ to the network, we assure that there exists a path between i and l, $P_{i;l}$, which is the special path. A line-star topology structure, which is the union of $P_{i;j}$ and $P_{i;l}$, is then found.

To demonstrate Algorithm 2, we apply it in an example. Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, which is illustrated in Fig. 11, contains three source nodes, $1, 2$ and 3. We note that there is no path from node 1 to node 2 meeting the condition that its deletion would not reduce the minimum cut of node 3, i.e., $\mathcal{C}_{3;1,2}$. Therefore, there are no rings in the graph. In the first step, we find a path between 1 and 2, $P_{1;2}$, whose deletion from the network reduces minimally the minimum cut of node 3, $\mathcal{C}_{3;1,2}$, e.g., $P_{1;2} = \{(1, 4), (4, 2)\}$. Next, we delete $P_{1;2}$ from the graph and then find all the remaining paths from node 3 to node 1 and delete them as well, e.g., $\{(3, 5), (5, 1)\}$. In the third phase, we restore path $P_{1;2}$ to the graph and search for a path from node 1 to node 3, e.g., $P_{1;3} = \{(1, 4), (4, 3)\}$. Finally, we conclude that the union of $P_{1;2}$ and $P_{1;3}$ is an element in $\mathcal{Q}$, i.e., $q = \{(1, 4), (2, 4), (3, 4)\}$, $q \in \mathcal{Q}$.

1) Complexity: To asses the complexity of Algorithms 1 and 2, it is important to note that they are both greedy in nature. That is, in Algorithm 1, for example, once a ring is found, it is deleted from the network for future iterations. There is no track-back or combinatorial search. The same holds for Algorithm 2 and line-star topologies. Thus, the complexity of these algorithms is bounded by $|\mathcal{R}|$ (or $|\mathcal{Q}|$) times the complexity of identifying one such object. Note that since one cannot surpass the min-cut outer bound, $|\mathcal{R}| + \frac{|\mathcal{Q}|}{2} \leq h$. Using only local characteristics, it is also not hard to see that $|\mathcal{R}| + |\mathcal{Q}| \leq \min_{i \in \mathcal{S}} deg(i)$, where $deg(i)$ is defined to be the degree of node $i \in \mathcal{V}'$. Hence, this multiplicative factor is at most linear.

Now, finding an element of $|\mathcal{R}|$ (or $|\mathcal{Q}|$) involves solving a flow problem between nodes in a graph. While it is more than a simple flow between two nodes, as one needs to verify other flows remain intact, since there are only finitely many (e.g., 3) sources and destinations, the problem remains polynomial in the size of the graph. For example, it is easy to see that finding an element of $\mathcal{R}$ is simply finding a cycle in an undirected graph (each bidirectional edge in our graph is replaced by one undirected edge) which passes through the three sources. This can be done in polynomial time [39]. Thus, for the problems at hand, complexity can remain polynomial in the size of the graph. In our simulations, we chose to solve this using the optimization problem (20). Such an optimization problem is not optimal in terms of complexity (it depends on the solver used), but proved useful in our simulations. Problems with about 150 nodes were solved in minutes using Matlab.

V. Multiple Unicast Network

In this section, we present a coding scheme for multiple unicast networks. Consider a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with a set of source nodes $\mathcal{S} = \{1, 2, 3\}$. In a multiple unicast network, each source node $i \in \mathcal{S}$ produces two different messages $W^{(t)}_{i \to j}$ and $W^{(t)}_{i \to l}$ that are intended for the two remaining source nodes $j, l \in \mathcal{S} \setminus \{i\}$, i.e., each two source nodes communicate bidirectionally in a unicast manner. Under the equal rate assumption, we show that the minimum cut upper bound is achievable. This should be compared to the multiple unicast problem in the general case, where the minimum cut upper bound is not tight [13]. Thus, our optimal coding scheme extends the current state of the art regarding what are the achievable rates. Next, we present the coding scheme for the corresponding broadcast model $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, depicted in Fig. 2b.

A. Capacity and Coding for a Multiple Unicast Network Based on Line Topology

Our coding scheme achieves the capacity region under an equal rate demand. Specifically, each two-way communication is carried out at the same rate, $R_{i \to j} = R_{j \to i}$, $\forall i, j \in \mathcal{S}$. To show the RT coding scheme for this network, we use the line topology coding scheme from Section III-A. Since each unicast session represents a flow from one source node to another, we show that no inter-flow coding is needed to achieve the equal rate capacity region. Our main result is summarized in the following theorem.

Theorem 5: For any multiple unicast network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ with three source nodes and an arbitrary integer delay bounded by D, there exists an RT coding scheme with any set of rates within the equal rate capacity region, which is, for all $i, j, l \in \mathcal{S}$,

$$\begin{aligned} R_{i \to j} + R_{i \to l} &\leq \mathcal{C}_{i;j,l}, \\ R_{i \to j} &\leq \mathcal{C}_{i;j}, \end{aligned} \tag{21}$$

where $R_{i \to j} = R_{j \to i}$ and $R_{i \to l} = R_{l \to i}$. Furthermore, the coding scheme includes a fixed header per transmission of $2\lceil \log_2 2D \rceil + \lceil \log_2 h \rceil$ bits.

Converse Proof of Theorem 5: The equal rate capacity region is upper bounded by standard minimum cut arguments. The first equation in (21) is derived from the minimum cut between node i and nodes j, l, while the second equation is derived from the minimum cut between nodes i and $j, l \in \mathcal{S}$. Corner points of $(R_{1 \to 2}, R_{1 \to 3}, R_{2 \to 3})$ in the rate region can be found by setting the rate $R_{1 \to 2} = \mathcal{C}_{1;2}$, which also yields that $R_{2 \to 1} = \mathcal{C}_{1;2}$. Therefore, $R_{1 \to 3} = \mathcal{C}_{1;2,3} - \mathcal{C}_{1;2}$ and $R_{2 \to 3} = \mathcal{C}_{2;1,3} - \mathcal{C}_{1;2}$. Hence, either $R_{2 \to 3}$ or $R_{1 \to 3}$ equals zero, since node 3 can only be in one of the cuts between nodes 1 and 2 and $\mathcal{C}_{1;2}$ is the minimum of all the cuts separating them:

$$\mathcal{C}_{1;2} = \min\{\mathcal{C}_{1;2,3}, \mathcal{C}_{2;1,3}\}. \tag{22}$$

Claim 1: The corner points of the equal rate capacity region described in (21) can be expressed as

$$\begin{aligned} R_{i\to j} &= \mathcal{C}_{i;j}, \\ R_{j\to l} &= [\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}]^+, \\ R_{i\to l} &= 0, \end{aligned} \tag{23}$$

where $[\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}]^+$ *is the maximum between* $\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}$ *and* 0*,* $i, j, l \in \mathcal{S}$*.*

The corner points can be found by setting $R_{i\to j} = \mathcal{C}_{i;j}$, $i, j \in \mathcal{S}$. Then, in the case where $\mathcal{C}_{i;j} = \mathcal{C}_{i;j,l}$, we get $R_{i\to l} = 0$ and $R_{j\to l} = \min\{\mathcal{C}_{j;l}, \mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}\}$, $l \in \mathcal{S} \setminus \{i, j\}$. Furthermore, the fact that $\mathcal{C}_{i;j,l} \le \mathcal{C}_{j;i,l} + \mathcal{C}_{l;i,j}$ implies (23).

Additionally, since the maximum distance in the graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ is L, we derive that the lower bound of the worst-case decoding delay is LD. This means that any coding scheme achieving the equal rate capacity cannot guarantee delay lower than LD. ■

To show an achievable coding scheme for the equal rate region in (21), it is sufficient to prove that we achieve all corner points in the rate region of the form of (23). Therefore, we present a lemma that shows how to partition each network $\mathcal{G}'$ into sub-networks of line topology. The decomposition includes a set of disjoint paths $\mathcal{P}_{i;j}$, where $|\mathcal{P}_{i;j}| = \frac{\mathcal{C}_{i;j,l}}{C}$, and a set of disjoint paths $\mathcal{P}_{j;l}$, where $|\mathcal{P}_{j;l}| = \frac{\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j,l}}{C}$ and $\mathcal{P}_{i;l} \cap \mathcal{P}_{j;l} = \emptyset$. Hence, by using this lemma, we can achieve the corner points in the equal rate capacity region.

Lemma 2: For a network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ *with three source nodes* i, j, l *and* $\mathcal{C}_{i;j} = \mathcal{C}_{i;j,l}$*, there exist sets of disjoint paths* $\mathcal{P}_{i;j}$ *and* $\mathcal{P}_{j;l}$ *such that* $|\mathcal{P}_{i;j}| + |\mathcal{P}_{j;l}| = \frac{\mathcal{C}_{j;i,l}}{C}$ *and* $|\mathcal{P}_{i;j}| = \frac{\mathcal{C}_{i;j,l}}{C}$*, where* $\mathcal{P}_{i;j} \cap \mathcal{P}_{j;l} = \emptyset$*. Namely,* $\mathcal{P}_{i;j}$ *and* $\mathcal{P}_{j;l}$ *have no mutual edges.*

Proof of Lemma 2: The maximum number of disjoint paths between nodes i and j, $|\mathcal{P}_{i;j}|$, is $\frac{\mathcal{C}_{i;j}}{C}$ (the Max-flow Min-cut theorem [40, Th. 1]). Since $\mathcal{C}_{i;j} = \mathcal{C}_{i;j,l}$, this is also the maximum number of paths between i and j, l. However, in the case where $\mathcal{C}_{i;j,l} < \mathcal{C}_{j;i,l}$, there are more paths between j and i, l than $\frac{\mathcal{C}_{i;j,l}}{C}$. These paths, $\mathcal{P}_{j;l}$, are between nodes j and l.

To prove that $\mathcal{P}_{i;j}$ and $\mathcal{P}_{j;l}$ are disjoint, we add $\frac{\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j,l}}{C}$ direct paths between i and l to the graph, $\mathcal{P}_{i;l}$, which yields a new graph in which $\mathcal{C'}_{i;j,l} = \mathcal{C}_{j;i,l}$, where $\mathcal{C'}_{i;j,l}$ is the minimum cut between i and j, l in the new graph. As a result of (22), we conclude that there are $\frac{\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j,l}}{C}$ new paths between nodes i and j, $\mathcal{P'}_{i;j}$. Paths $\mathcal{P'}_{i;j}$ are a union of $\mathcal{P}_{i;l}$ and $\mathcal{P}_{j;l}$, i.e., $|\mathcal{P}_{j;l}| = \frac{\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j,l}}{C}$. Hence, since $\mathcal{P}_{i;j}$ and $\mathcal{P'}_{i;j}$ are disjoint according to [40], we conclude that $\mathcal{P}_{j;l}$ and $\mathcal{P}_{i;j}$ are also disjoint. ■

Next, we provide the achievability proof for Theorem 5.

Achievability Proof of Theorem 5: Using the line topology coding scheme from Section III-A and Lemma 2, we can achieve all the corner points in the capacity region. Specifically, we use the line topology coding scheme from Section III-A at each path constructed via Lemma 2 to achieve a rate of $R_{i\to j} = \mathcal{C}_{i;j}$ and $R_{j\to l} = \mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}$, $i, j, l \in \mathcal{S}$. Using time sharing arguments, it is straightforward to see that one can achieve the capacity region under the equal rate constraint. Moreover, this coding scheme achieves the minimum worst-case decoding delay of $LD + t$, by using a header of $2\lceil \log_2 2D \rceil$ bits (according to the coding scheme of the line topology coding scheme from Section III-A). Another $\lceil \log_2 h \rceil$ bits are used to distinguish between the disjoint line sub-networks. Therefore, we obtain an RT NC scheme that achieves the equal rate capacity. ■

B. Algorithms

We now introduce an optimization problem that can be used to find the set of paths predicted in Lemma 2, i.e., $\mathcal{P}_{i;j}$ and $\mathcal{P}_{j;l}$, $i, j, l \in \mathcal{S}$. In this problem, we would like to find the minimum flow, $\mathbf{f}$, which satisfies the condition that $|\mathcal{P}_{j;l}| = \frac{\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}}{C}$ paths from node j to node l and another $|\mathcal{P}_{i;j}| = \frac{\mathcal{C}_{i;j}}{C}$ paths from node i to node j are disjoint. Since we would like to find $\frac{\mathcal{C}_{i;j}}{C}$ paths between nodes i and j, we demand that $\mathcal{I}_i = \frac{\mathcal{C}_{i;j}}{C}$ and $\mathcal{O}_j = \frac{\mathcal{C}_{j;i,l}}{C}$. However, for the case in which $\mathcal{C}_{i;j} < \mathcal{C}_{j;i,l}$, there are more paths emerging from node j. Those paths are consumed by node l, i.e., $\mathcal{I}_l - \mathcal{O}_l = \frac{\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}}{C}$. We guarantee that all of the paths are disjoint since $\mathbf{f}$ is a binary vector and each relay node in the network, $m \in \mathcal{V}' \setminus \mathcal{S}$, has equal input and output flows. Hence, the outcome of this optimization problem is a binary flow that includes two sets of disjoint paths $\mathcal{P}_{i;j}$ and $\mathcal{P}_{j;l}$:

$$\begin{aligned} &\underset{\mathbf{f}}{\text{minimize}} \sum_{f_{(u,v)} \in \mathbf{f}} f_{(u,v)} \\ &\text{subject to } \mathcal{O}_m = \sum_{n:(m,n)\in\mathcal{E}'} f_{(m,n)}, \quad m = 1, \ldots, |\mathcal{V}| \\ &\qquad \mathcal{I}_m = \sum_{n:(n,m)\in\mathcal{E}'} f_{(n,m)}, \quad m = 1, \ldots, |\mathcal{V}| \\ &\qquad \mathcal{I}_j = 0 \\ &\qquad \mathcal{I}_i = \frac{\mathcal{C}_{i;j}}{C} \\ &\qquad \mathcal{I}_l - \mathcal{O}_l = \frac{\mathcal{C}_{j;i,l} - \mathcal{C}_{i;j}}{C} \\ &\qquad \mathcal{O}_j = \frac{\mathcal{C}_{j;i,l}}{C} \\ &\qquad \mathcal{O}_i = 0 \\ &\qquad \mathcal{O}_k = \mathcal{I}_k, \quad k = 4, \ldots, |\mathcal{V}|, \end{aligned} \tag{24}$$

where $i, j, l \in \mathcal{S}$.

C. Example

In this subsection, we show an example of a network $\mathcal{G}$ with three source nodes that communicate in a bidirectional unicast manner (Fig. 12). In fact, we provide an example in which we apply the algorithm from the previous Subsection V-B, i.e., the algorithm for finding the disjoint paths between the three source nodes that, by using the line topology coding scheme from Section III-A, achieves a corner point in the equal rate capacity region. The corner point that is shown in Figs. 12b and c is $(R_{1\to2}, R_{1\to3}, R_{2\to3}) = (\mathcal{C}_{1;2,3} - \mathcal{C}_{1;3}, \mathcal{C}_{1;3}, 0)$. Specifically, in this network $\mathcal{C}_{1;3} = 3C$ and $\mathcal{C}_{1;2,3} = 5C$ and, therefore, we show three disjoint paths

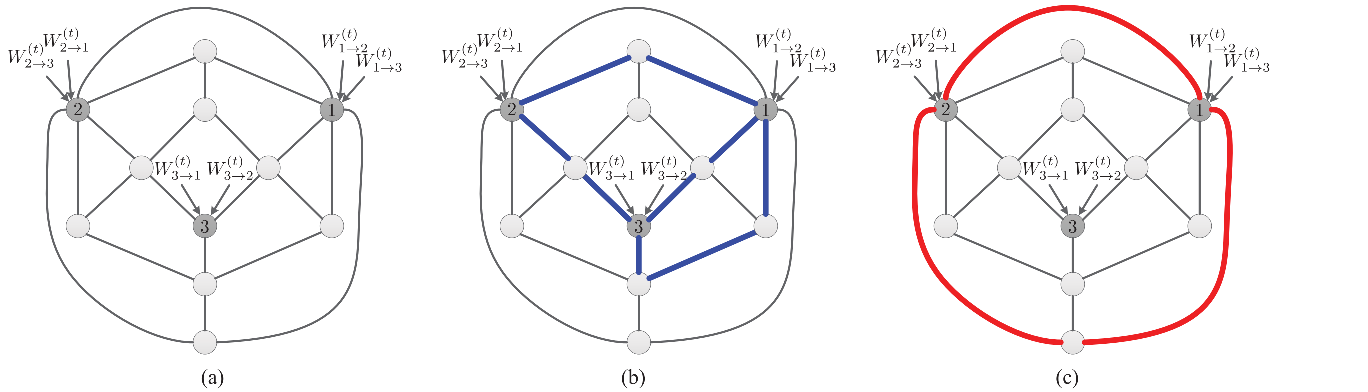

Fig. 12. A network $\mathcal{G}$ is shown in (a), where all the source nodes $1, 2$ and 3 communicate in a bidirectional unicast manner. In (b) and (c), we show a corner point in the equal rate capacity region by applying the algorithm from Section V-B, where $R_{1\to3} = 3C$, and $R_{1\to2} = 2C$.

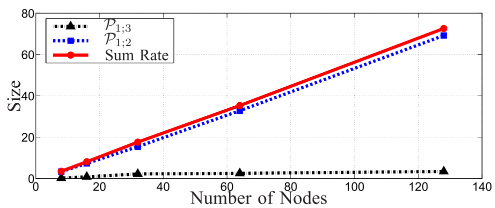

Fig. 13. Simulation results of the average size of the sets $\mathcal{P}_{1;2}$ and $\mathcal{P}_{1;3}$ in a multiple unicast network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$. This simulation shows that the corner point of $R_{1\to2} = \mathcal{C}_{1;2}, R_{1\to3} = \mathcal{C}_{1;2,3} - \mathcal{C}_{1;2}$ and $R_{2\to3} = 0$.

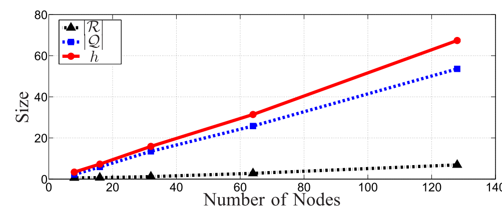

Fig. 14. Simulation results of the average sizes of the sets $\mathcal{R}$ and $\mathcal{Q}$ and the upper bound h in a multicast network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$. Recall that $2\mathcal{R} + \mathcal{Q} \geq h$.

from node 1 to node 3 (Fig. 12b) and another two disjoint paths from node 1 to node 2 (Fig. 12c). The paths are disjoint since they do not pass through a common relay node; similarly, in the corresponding graph $\mathcal{G}'$ they have no mutual edges.

VI. Simulation

The research work reported so far has been analytical in nature; to gain more intuition and insight we conducted simulations using a fixed number of nodes and a random number of edges on an Erdős-Rényi random graph model [41]. In this model, each edge is a bidirectional link that is drawn independently of the other links with probability p to have capacity C or $(1-p)$ to have capacity 0. Then, we constructed an equivalent graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, as illustrated in Fig. 3, and searched for the line and star sub-networks in the graphs using Algorithms 1 and 2.

Fig. 13, depicts the results for finding a corner point in the equal rate capacity region of a multiple unicast network as described in Section V. This corner point is the result of a maximization on the rate between nodes 1 and 2, i.e., $R_{1\to2} = \mathcal{C}_{1;2}$. Then, we search for the maximum number of disjoint paths between nodes 1 and 3, i.e., $R_{1\to3} = \mathcal{C}_{1;2,3} - \mathcal{C}_{1;2}$. Therefore, the number of paths between nodes 1 and 2 is likely to be greater than the number of paths between nodes 1 and 3. For each simulation, we compared 8, 16, 32, 64 and 128 nodes by creating ten different graphs for each number of nodes and varying the value of p. We show the average number of $|\mathcal{P}_{1;2}|$ and $|\mathcal{P}_{1;3}|$ over the number of nodes in a graph $\mathcal{G}$. As expected, our result shows that each of the examined variables ($|\mathcal{P}_{1;2}|$ and $|\mathcal{P}_{1;3}|$) increases with the scale of the network, where $|\mathcal{P}_{1;2}| > |\mathcal{P}_{1;3}|$. Furthermore, this simulation shows that $\mathcal{P}_{1;2}$ is more significant than $\mathcal{P}_{1;3}$, i.e., $\mathcal{C}_{1;2} \gg \mathcal{C}_{1;2,3} - \mathcal{C}_{1;2}$. Namely, in a multiple unicast network of three source nodes, using time sharing and only one bidirectional unicast at a time achieves a rate that is close to optimal. For example, instead of the corner point shown in Fig. 13, using the corner point of $R_{1\to2} = \mathcal{C}_{1;2}$ and $R_{1\to3} = R_{2\to3} = 0$ is close to optimal.

We further performed simulations on a network with the multicast demands from Section IV, illustrated in Fig. 14. In this network, we searched for the maximum number of ring (Algorithm 1) and line-star (Algorithm 2) sub-networks in the graph. This simulation shows the decomposing opportunities in a graph. We recall that each ring sub-network contributes a rate of C to each source, where each line-star sub-network only contributes $\frac{C}{2}$. However, we see from Fig. 14 that the line-star sub-network is more dominant than the ring sub-network in the multicast problem of three users under the equal rate constraint. This is because in a large scale network there are not a lot of paths between two source nodes whose deletion from the graph does not reduce the minimum cut of the third source node.

In the next simulation (see Figs. 15 and 16) we compared the performance of our coding schemes to simple store-and-forward schemes in a network, $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Specifically, we measured the number of transmissions required by the relay nodes, $\mathcal{V} \setminus \mathcal{S}$, in the coding schemes that achieves the maximum rate for each network. The simulation shows a marked improvement in the number of required transmissions due to the efficient use of broadcast transmission. Unlike a simple store-and-forward technique, in the proposed coding schemes we employ NC to allow at least two adjacent nodes to gain new information from each transmission. The differences between

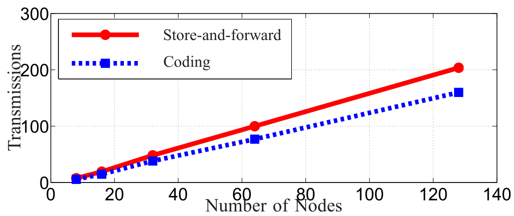

Fig. 15. Simulation results of the average number of transmissions required by the relay nodes in a multicast network. As shown, we achieve better performance using the proposed coding scheme.

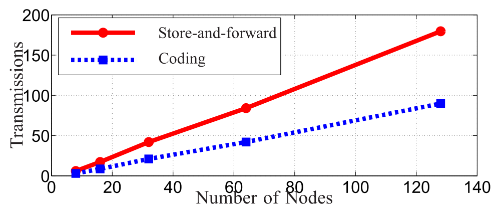

Fig. 16. Simulation results of the average number of transmissions required by the relay nodes in a multiple unicast network. As shown, we achieve significantly better performance using the proposed coding scheme.

the multicast (Fig. 15) and the multiple unicast (Fig. 16) scenarios are due to the differences between the star topology coding scheme (Section III-B), which makes a more significant contribution in the multicast network, and the line topology coding scheme (Section III-A), which is the only sub-network that is used in the multiple unicast network. Therefore, the improvement for the multiple unicast network is 100%, and for the multicast network is about 30%.

VII. Extensions

In this section, we show that the building block approach also yields a coding scheme that achieves capacity for the combined problem, namely, a network of multicast and multiple unicast demands. By using the coding schemes for the multicast (Section IV) and the multiple unicast (Section V) scenarios, we show that such a combination is feasible, i.e., we get a coding scheme achieving capacity under the equal rate demand. Additionally, we discuss the difficulties encountered in the case of a network with four source nodes. Specifically, we show that the line and line-star building blocks approach that was presented in the previous sections is no longer sufficient to achieve the capacity region in the corresponding broadcast model.

A. A Network With Multicast and Multiple Unicast Demands

Here, we use the line and line-star topology coding schemes to show that there exists an RT coding scheme for a general network, $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, in which three source nodes communicate bidirectionally in multicast and multiple unicast manners and which achieves the equal rate capacity region (Fig. 17). Our coding scheme achieves the capacity region under an equal rate demand. Specifically, each two-way communication is carried out at the same rate, $R_{i\to j} = R_{j\to i}$ and $R_i = R_j$, $\forall i, j \in \mathcal{S}$. Each source node $i \in \mathcal{S}$ produces three different messages, $W_i^{(t)}, W_{i\to j}^{(t)}$ and $W_{i\to l}^{(t)}$, which are intended for the two remaining source nodes $j, l \in \mathcal{S} \setminus \{i\}$.

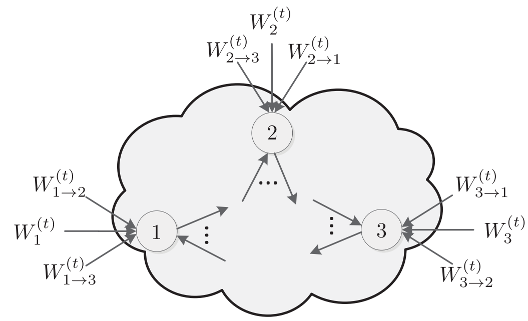

Fig. 17. Schematic illustration of the network, where $W_i^{(t)}$ is a message that node i generates at time instant t and that is destined for all the other source nodes, and $W_{i\to j}^{(t)}$ is a message that node i generates at time instant t, and it is only destined for node j.

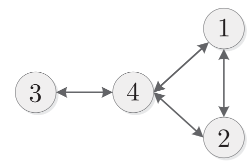

Fig. 18. Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{S} = \{1, 2, 3\}$.

Corollary 1: For any network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ with three source nodes and an arbitrary integer delay bounded by D, there exists an RT coding scheme with any set of rates within the equal rate capacity region, which is

$$\begin{aligned} 2R_i + R_{i\to j} + R_{i\to l} &\le \mathcal{C}_{i;j,l} \\ R_{i\to j} &\le \mathcal{C}_{i;j}, \end{aligned} \tag{25}$$

where $R_i = R_j$, $R_{i\to j} = R_{j\to i}$ and $R_{i\to l} = R_{l\to i}$, $\forall i, j, l \in \mathcal{S}$. Furthermore, the coding scheme includes a fixed header per transmission of $2\lceil \log_2 2D \rceil + \lceil \log_2 h \rceil$ bits.

Proof: The equal rate capacity region can be upper bounded by evoking standard minimum cut arguments. For example, $\mathcal{C}_{1;2,3}$ represents the cut set bound of all the information that node 1 receives. Nodes 2 and 3 transmit multicast and unicast messages to node 1, e.g., $W_2^{(t)}, W_3^{(t)}, W_{2\to 1}^{(t)}$ and $W_{3\to 1}^{(t)}$. Therefore, we obtain the first upper bound in the region (25). The second upper bound is obtained straightforwardly by the minimum-cut maximum-flow theorem.

To show an achievable coding scheme for the region in (25), it is sufficient to prove that we achieve all corner points in the rate region. There exist two different corner points: the first is the case $R_{i\to j} = \mathcal{C}_{i;j}$, which yields that $R_i = 0$ since

$$\mathcal{C}_{i;j} = \min\{\mathcal{C}_{i;j,l}, \mathcal{C}_{j;i,l}\}, \tag{26}$$

where $i, j, l \in \mathcal{S}$. Therefore, this corner point is strictly the multiple unicast case, which we already discussed in Section V. The second corner point is the case where $R_i = \frac{h}{2}$, $\forall i \in \mathcal{S}$. In this case, there are two bidirectional unicast sessions that equal zero. For example, assuming $h = \mathcal{C}_{1;2,3}$ yields that $R_{1\to 2}$ and $R_{1\to 3}$ both equal zero. The remaining unicast rate, e.g., $R_{2\to 3}$, can be larger than zero in the general case, i.e., we can find a set of disjoint paths between nodes 2 and 3 such that their deletion from

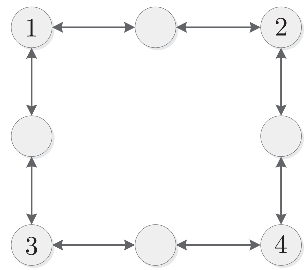

Fig. 19. Multicast network $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with four source nodes, where a coding scheme based on the line and line-star building blocks does not achieve the equal rate upper bound.

the network would not reduce the maximum rate of the multicast session. For example, consider the network depicted in Fig. 18. In this network, there exists a path between nodes 1 and 2 that does not intersect with the star topology sub-network, which is required to obtain the maximum multicast rate (in this case $\frac{C}{2}$). ■

B. A Network With Four Source Nodes

In this subsection, we would like to discuss the case of a network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ with four source nodes, where using the line and line-star building blocks approach failed to achieve the equal rate capacity.

In the following network, nodes $1, 2, 3$ and 4 communicate in a multicast manner (Fig. 19). The equal rate upper bound is $R_i \leq \frac{2}{3}C$, $i \in \mathcal{S}$, as can be obtained by evoking the standard minimum cut arguments. Without using the proposed building block approach, for example, this upper bound can be achieved by using the line topology coding scheme in Section III-A from node 1 to node 2 and from node 1 to node 3, i.e., the message $W_1^{(t)}$ (with length $\frac{2}{3}C$) is sent to nodes 2 and 3. Then, nodes 2 and 3 each can send half of the message, $W_1^{(t)}$, to node 4 by using the line topology coding scheme from node 2 to node 4 and between node 3 and node 4. Thus, using this coding scheme achieves the maximum rate.

However, since there is no path from node 1 to node 4 whose deletion from the graph would not reduce the minimum cut of node 2, $\mathcal{C}_{2;1,3,4}$, or the minimum cut of node 3, $\mathcal{C}_{3;1,2,4}$, there are no rings in the network. Therefore, to achieve the equal rate upper bound, two separate star topologies have to be found. However, there is only one, i.e., a rate of $\frac{1}{3}C$ is achieved. Hence, by using the building blocks of ring and line-star topologies in a network with more than three users, the equal rate upper bound is unreachable in the general case. Similarly, the ring and line-star building blocks approach also fails to achieve the equal rate capacity region for more than three users in the multiple unicast case, in which the minimum cut upper bound is not tight. Nevertheless, we expect that although our building blocks approach is insufficient in this case, another combination of building blocks will achieve the equal rate upper bound.

VIII. Conclusion

We conclude that a network with three source nodes under any communication demands can be decomposed into line and line-star topologies using the building block approach. Then, by exploiting the broadcast ability of the wireless medium, we can achieve the capacity region under an equal rate constraint in the corresponding broadcast model. Furthermore, the coding schemes, which are based on those two canonical topologies, display many advantages, such as simple decoding algorithms, RT decoding delay, small overhead and error propagation. In practice, these coding schemes can be implemented on wireless networks with arbitrary transmission delays to gain better performance and power efficiency.

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