Final Exam Solution - Moed A

1 True or False:

(a) True

$$H(X,Y,Z) - I(X;g(X)) = H(X,Y,Z) - H(g(X)) + \cancel{H(g(X)|X)}^{\to 0}$$

$$= H(X,Y,Z,g(X)) - H(g(X)) = H(X,Y,Z|g(X))$$

(b) (i) True

$$h(X|Y) = h(X-f(Y)|Y) \quad \forall \text{ function } f(\cdot)$$

so $h(X|Y) = h(X - \underbrace{\hat{X}^{lin}}_{f(Y)}|Y)$

(ii) True

For a jointly Gaussian RVs the optimal linear estimator is also the optimal estimato for which orthogonality principle hold, i.e.

$\mathcal{E} = X - \hat{X}^{lin} \perp\!\!\!\perp g(Y)$ for every $g(\cdot)$

Thus:

$$h(X-\hat{X}^{lin}|Y) \overset{X-\hat{X}^{lin} \perp\!\!\!\perp Y}{=} h(X-\hat{X}^{lin})$$

(c) (i) True

(ii) True

(iii) False

(d) False

If $f(x)$ is convex so by the perspective transform $g(t,x) = t \cdot f(\frac{x}{t})$ is convex for $t \geq 0$ and not convex for $t<0$.

Therefore a linear transformation will also not be convex for $t<0$

② Channel with State:

(a) Denote the capacity of the S-Channel by

$$C_S = \max_{p(x|s=1)} I(X;Y|S=1)$$

$$= \max_{p(x|s=1)} \Big[\underbrace{H(Y|S=1)}_{(*)} - \underbrace{H(Y|X,S=1)}_{(**)} \Big]$$

Assume that the input X is distributed according to $X \sim$ Bernuli (α) $(p(X=0)=\alpha,\ p(X=1)=1-\alpha)$ for $S=1$. and let us calculate the entropy terms:

$$(**) = H(Y|X,S=1) = \sum_{x\in\mathcal{X}} p(x) H(Y|X=x,S=1)$$

$$= \alpha H(Y|X=0,S=1) + (1-\alpha) H(Y|X=1,S=1)$$

$$= \alpha H_b(\varepsilon) + \cancel{(1-\alpha)\cdot 0}^{\to 0} = \alpha H_b(\varepsilon)$$

$$(*) = H(Y|S=1)$$

$$p(y=0|S=1) = p(y=0,x=0|S=1) + p(y=0,x=1|S=1)$$

$$= p(x=0|S=1)\,p(y=0|x=0,s=1) + p(x=1|S=1)\,p(y=0|x=1,s=1)$$

$$= \alpha\cdot(1-\varepsilon) + \cancel{(1-\alpha)\cdot 0}^{\to 0} = \alpha(1-\varepsilon)$$

$$p(y=1|S=1) = 1-\alpha(1-\varepsilon)$$

$$H(Y|S=1) = H_b(\alpha(1-\varepsilon))$$

In order to find the capacity we differentiate

$$I(X;Y|S=1) = H_b(\alpha(1-\varepsilon)) - \alpha H_b(\varepsilon)$$

with respect to α and find the roots of the derivation

$$\frac{\partial}{\partial \alpha} I(X;Y|S=1) = \frac{\partial}{\partial \alpha}\left[H_b(0.8\alpha) - \alpha H_b(0.2)\right] = 0$$

$$\Rightarrow \boxed{\alpha^* \simeq 0.435664}$$

$$\boxed{C_S = I(X;Y|S=1)\Big|_{\alpha=\alpha^*} = 0.618231} \quad \left[\frac{\text{bits}}{\text{channel use}}\right]$$

(b) For the binary symmetric channel (BSC) the capacity is given by:

$$C_{BCS} = I(X;Y|S=2) = 1 - H_b(\delta)$$

Taking $\delta = 0.1$

$$\boxed{C_{BSC} = 1 - H_b(0.1) = 1 - 0.468996 = 0.531} \quad \left[\frac{\text{bits}}{\text{channel use}}\right]$$

(c) The capacity of the Z-Channel and the S-channel are equal since the channels are equivalent up to switching 0 with 1 and vice versa we have:

$$\boxed{C_Z = I(X;Y|S=3) = C_S = I(X;Y|S=1) = 0.618231} \left[\frac{\text{bits}}{\text{channel use}}\right]$$

(d) The capacity of a channel with states:

$$C = \max_{p(x|s)} I(X;Y|S)$$

$$= \max_{p(x|s)} \left[p(S=1) I(X;Y|S=1) + p(S=2) I(X;Y|S=2) + p(S=3) I(X;Y|S=3)\right]$$

$$= \frac{1}{2} C_S + \frac{1}{3} C_{BCS} + \frac{1}{6} C_Z$$

$$= \boxed{0.589154}$$

(e) The rows of the matrix $P_{X|S}$ are the probability function which achieve capacity for each of the sub channel. i.e:

$$P_{X|S} = \begin{bmatrix} p(x=0|S=1) & p(x= \;\; |S=1) \\ p(x=0|S=2) & p(x=1|S=2) \\ p(x=0|S=3) & p(x=1|S=3) \end{bmatrix} = \boxed{\begin{bmatrix} 0.435 & 0.565 \\ 0.5 & 0.5 \\ 0.565 & 0.435 \end{bmatrix}}$$

where we have used the fact that the input probability that achieves capacity for the BSC is $[0.5 \quad 0.5]$ (i.e Bernouli $(\frac{1}{2})$)

③ (a) We need to find a row vector μ such that $\mu \cdot P = \mu$ holds. Denote $\mu = [q \quad 1-q]$

(First equation)

$$\mu P = \mu \Rightarrow q(1-a) + 1(1-q) = q \Rightarrow q = \frac{1}{1+a}$$

Thus: $$\boxed{\mu = \left[\frac{1}{1+a} \quad \frac{a}{1+a} \right]}$$

(b) We need to calculate $\lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \dots, X_n)$

$$\lim_{n\to\infty} \frac{1}{n} H(X_1^n) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i | X^{i-1})$$

Chain rule

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i | X_{i-1})$$

Markov property $X_i - X_{i-1} - X_1^{i-2}$ for all $i \geq 1$ where $X_j = 0$ for $j \leq 0$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[p(X_{i-1}=0) H(X_i | X_{i-1}=0) + p(X_{i-1}=1) H(X_i | X_{i-1}=1) \right]$$

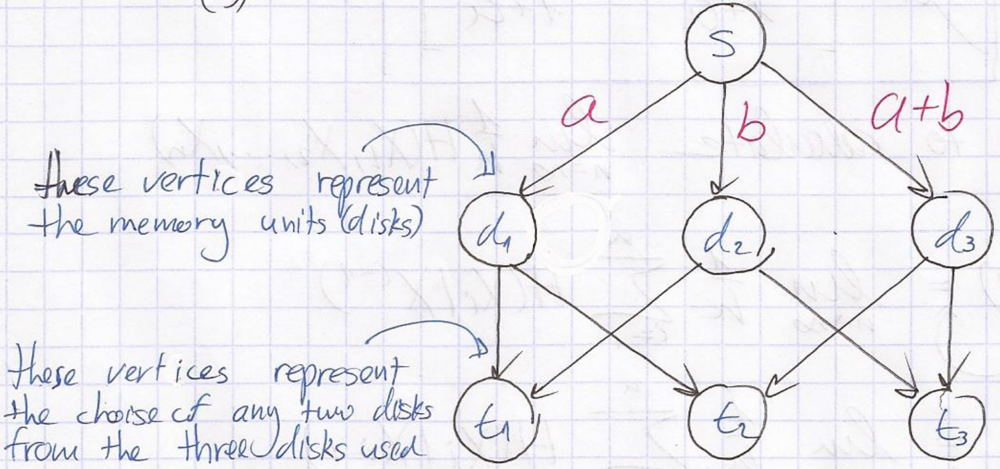
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} q \cdot H(a) + (1-q) \cdot 0$$

$$= \frac{H(a)}{1+a}$$

$\Rightarrow$ The entropy rate $\boxed{H(\mathcal{X}) = \frac{H(a)}{1+a}}$

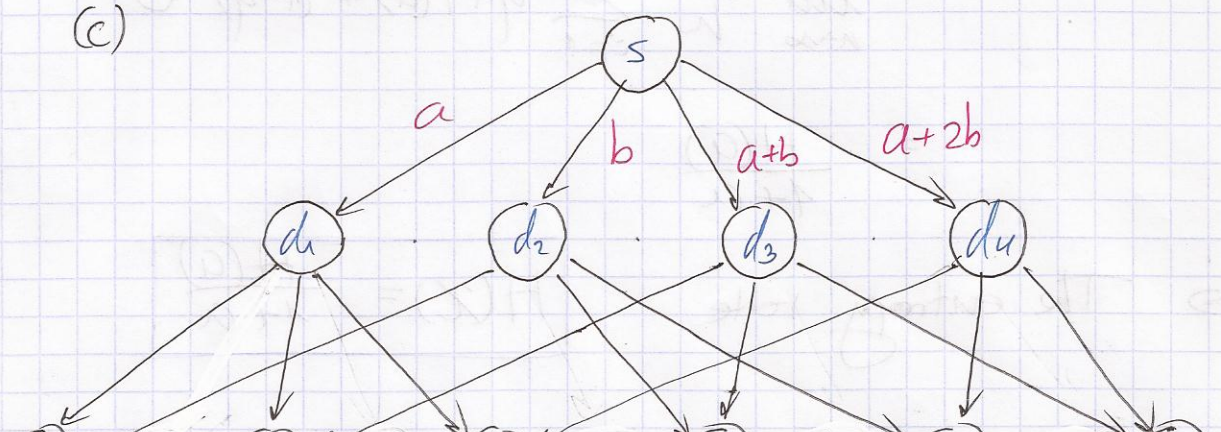
(4) (a) Since we need to reconstruct the data given any two hardisks and the source is distributed according to Berouli ($\frac{1}{2}$) (i.e. no further compression is possible) each of the three hardisks must be at least of size $\boxed{6\text{ Gbit}}$

(b)



Since mincut = 2 in this scheme it is possible to transfer two bits using this scheme (although each memory unit (disk) recieves only one bit). The achievable scheme is: $\boxed{\text{In red.}}$

(c)



To solve this network coding problem we need to find 4 vectors which are pairwise independent. This cannot be done using a binary alphabet ($|\mathbb{F}|=2$) but it is possible using a ternary alphabet ($|\mathbb{F}|=3$). The achievable scheme: $\boxed{\text{In red}}$