Introduction to Information and Coding theory 5 Aug 2012
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Final Exam (Moed B)

1) **True or False**
Copy each relation to your notebook and write **true** or **false**. Then, if it's true, prove it. If it is false give a counterexample or prove that the opposite is true.
   a) Let X, Y, Z be three random variables, with the Markov relation $X - Y - Z$. Then $H(X, Z) \geq H(X|Y) + H(Z|Y)$.
   b) Let $X \sim Ber(\alpha)$ be a random variable. For any function g, $H(X) = H(X - g(X))$.
   c) Let X, Y, Z be three random variables, with the Markov relation $X - Y - Z$. Then $I(X; Z|Y) \geq H(X, Z|Y) - H(Z|Y)$.
   d) Let $f(x)$ be a concave function, $g(x)$ is a convex non-increasing function. Is the function $h(x) = g(f(x))$ convex? (you may assume that the second derivative exists)
2) **Additive Gaussian noise**
In this question we consider a channel with additive Gaussian noise as seen in class.
Consider the channel presented in Fig. 1.

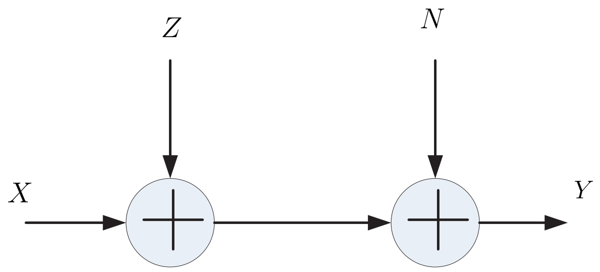

Fig. 1. Additive Gaussian noise channel.

$$Y = X + Z + N,$$

where $Z \sim \mathcal{N}(0, \sigma_1^2)$ and $N \sim \mathcal{N}(0, \sigma_2^2)$ are additive noises and the input, X, is with power constraint P. N, Z and X are independent.
   a) Calculate the capacity of the channel assuming that the noise is independent of the message that the encoder uses for determining X_i.
   b) Now it is given that Z_i is an output of a relay-encoder which has access to the same message M that the channel encoder has. Hence X and Z are no longer independent. It is also given that Z has a power constraint P, namely $\frac{1}{n}\sum_{i=1}^{n} Z_i \leq P$ with high probability. Find the capacity of the channel and the probability density function $f(x, z)$ for which it is achieved.
3) **Differential entropy upper bound where the covariance matrix is fixed**
Let X, Y be two random variable with continuous alphabet that is the real line $\mathbb{R}$. The random variables have fixed covariance matrix $\Sigma_{X,Y}$ and mean zero. Assume the determinant of $\Sigma_{X,Y}$ is positive and the probability density function $f(x, y)$ exists for any $x, y \in \mathbb{R}$. State which of the differential entropies expression are upper-bounded and in case they are upper-bounded express the bound using $\Sigma_{X,Y}$.
   a) $h(X)$
   b) $h(X, Y)$
   c) $h(X|Y)$

4) **drawing a codebook** Let X_i be a r.v. i.i.d distributed according to $P(x)$. We draw codebook of 2^{nR} codewords of X^n independently using $P(x)$ and i.i.d.. We would like to answer the question: what is the probability that the first codeword would be identical to another codeword in the codebook as n goes to infinity.
   a) Let x^n be a sequence in the typical set $A_\epsilon^n(X)$. What is the asymptotic probability (you may provide an upper and lower bound) as $n \to \infty$ that we draw a sequence X^n i.i.d distributed according to $P(x)$ and we get x^n.
   b) Using your answer from the previous sub-question find an $\overline{\alpha}$ such that if $R < \overline{\alpha}$ the probability that the first codewaord in the codebook appears twice or more in the codebook goes to zero as $n \to \infty$.
   c) Find an $\underline{\alpha}$ such that if $R > \underline{\alpha}$ the probability that the first codewaord in the codebook appears twice or more in the codebook goes to 1 as $n \to \infty$.
5) **Network coding with state** Assume we have a network-coding problem as studied in class characterized by $(\mathcal{N}, \mathcal{E})$ where the capacity links changes over time i according to some binary state Z_i. Namely, if at time i the state $Z_i = 0$ then the capacity links are $\mathcal{C}_0$ and if at time i the state $Z_i = 1$ then the capacity links are $\mathcal{C}_1$. The network is used once each time and no delay exists at the nodes. The set of vertices $\mathcal{N}$ and the set of edges $\mathcal{E}$ does not change. The state Z_i is distributed i.i.d. according to $P(z)$ and is known to all the nodes non-causally, namely at time i, Z^n (where $n > i$) is known to all the nodes including the source nodes, the relay nodes and the destination nodes. Therefore the code may depends on the states up to time i, i.e., Z^i and the capacity is defined as in class over many usages of the network (not just one use). The state sequence Z^n is independent of the sources sequences.
   Hint: The idea of state is similar to the cases we studied in class of losless source coding with states or channel with states that we saw in class and saw in the HW.
   a) Provide a mathematical definition of the code and capacity of the network for such a problem.
   b) For the problem of one-source one destination, what is an upper bound on the capacity? (provide proof)
   c) Provide a coding scheme of achieving this upper bound. You may use as a black box coding scheme used in class.
   d) What is the capacity for the case of one-source multi-destination for the network with states. (no proof needed here).
   e) Would you answer changes if the state where known causally?

Good Luck!