Introduction to Information and Coding theory 30 June 2013
(Dr. Permuter Haim and Mr. Ziv Goldfeld)

Final Exam

Total time for the exam: 3 hours!

1) **True or False** (30 points):
Copy each relation to your notebook and write **true** or **false**. Then, if it's true, prove it. If it is false give a counterexample or prove that the opposite is true.
   a) Let X be a continuous random variable. Then the following holds
$$I(X;X) = h(X).$$
   b) Let X, Y, Z be three random variables that satisfies $H(X,Y) = H(X) + H(Y)$ and $H(Y,Z) = H(Z) + H(Y)$. Then the following holds
$$H(X,Y,Z) = H(X) + H(Y) + H(Z).$$
   c) For any X, Y, Z and the deterministic function f, g
$$I(X;Y|Z) = I(X, f(X,Y); Y, g(Y,Z)|Z).$$
   d) $H(X|Z)$ is concave in $P_{X|Z}$ for fixed P_Z.
   e) Let $P(y|x)$ characterize a channel with Binary input and let $P(y|x=1) = P(y|x=0)$ for all $y \in \mathcal{Y}$. The capacity of this channel is 0.
2) **Two antennas with Gaussian noise** (20 points): In this question we consider a point-to-point discrete memoryless channel (DMC) in which the transmitter and the receiver both have two antennas, illustrated in Fig. 1. This channel is defined by two input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, two output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and a channel transition matrix $P_{Y_1Y_2|X_1X_2}$. A message M is randomly and uniformly chosen from the message set $\mathcal{M} = \{1, 2, \ldots, 2^{nR}\}$ and is to be transmitted from the encoder to the decoder in a lossless manner (as defined in class).

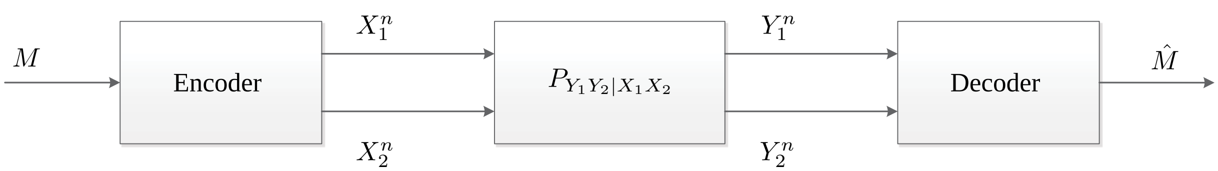

Fig. 1. Two antenna point-to-point DMC.

   a) What is the capacity of the channel?

Now, consider the following Gaussian two antenna point-to-point DMC illustrated in Fig. 2

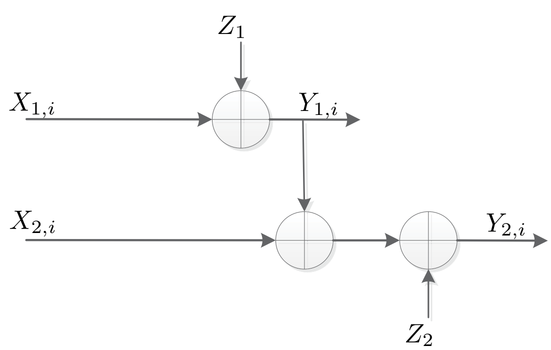

Fig. 2. A Gaussian two antenna point-to-point DMC.

The outputs of the channel for every time $i \in \{1, \ldots, n\}$ are give by,
$$Y_{1,i} = X_{1,i} + Z_1, \quad (1)$$
$$Y_{2,i} = X_{1,i} + X_{2,i} + Z_1 + Z_2, \quad (2)$$
where (Z_1, Z_2) are two independent (of each other and of everything else) Gaussian random variable distributed according to $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z_1 \sim \mathcal{N}(0, N_2)$. The input signals are bound to an average power constraints,
$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_{1,i}^2\right] \le P_1 \quad ; \quad \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_{2,i}^2\right] \le P_2. \quad (3)$$

b) Find the capacity of the Gaussian channel in terms of the provided parameters and state the joint distribution of (X_1, X_2) that achieves it.

3) **Riddle** (20 points): In a magic trick, there are three participants: the magician, an assistant, and a volunteer. The assistant, who claims to have paranormal abilities, is in a soundproof room. The magician has a deck of 100 cards. A different number from 1 to 100 is written on each card (in other words, they are numbered from 1 to 100). The magician asks a volunteer from the crowd to pick six cards. Then, the cards are shown to the crowd. The volunteer keeps one of the cards. The magician arranges the five cards that are left in some order. Now the assistant comes to the stage looks at the five cards and announces the number of the card kept by the volunteer!

a) The magician and the assistant are experts in information theory. How did they preform the trick? (Hint: One can used the 5 remaining cards to encode a message).

b) Can the magician and the assistant preform this trick with more then 100 cards? If yes, explain how many. If the answer is no, explain.

4) **Network coding** (30 points): Consider the network shown in Fig.3. This network consists of one source node S, two destination nodes D_1, D_2, 12 another nodes and a XOR gate (the most left circle). Every edge (i, j) in the network represents a directional noiseless link from node i to j, that can transmit 1 bit per second.

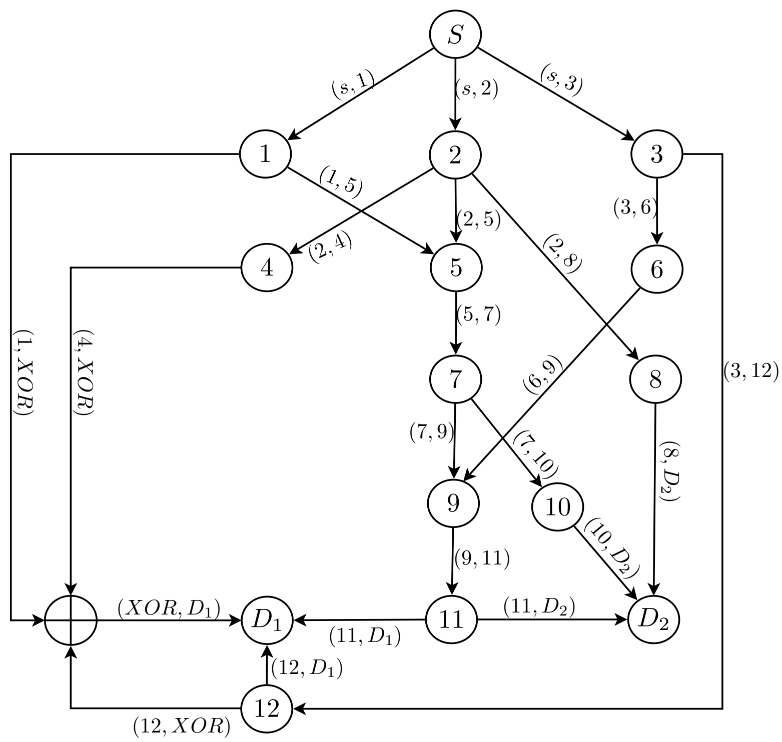

Fig. 3. A network with one source node, two destination nodes, another 12 nodes and a XOR gate

In subquestions a-c the XOR gate is replaced by an ordinary node 13 like the other nodes $1-12$ (it doesn't have to do a XOR operation on the received bits).

a) For every destination node, find how many edge-disjoint paths exist between the source node and that destination node?

b) We want to transmit data from the source node to the destination node D_i ($i = 1$ or 2). The other destination node doesn't have to receive the data. What is the maximal transmission rate R_i (in bits per second) that can be achieved in that case? Find for $i = 1, 2$.

c) Can we achieve these rates by simple routing scheme, where every node sends on its output links only bits there were received at its input links.

d) Design a linear network code that would allow the source node to transmit data to both destination nodes D_1, D_2 at the rates R_1, R_2. (The XOR node behave in this subsection as XOR, namely the output is a XOR of all inputs.) Draw the network and on each edge write the linear function that is applied.

e) Write the transfer matrix for each destination as a function of the source.

Good Luck!