Introduction to Information and Coding Theory July 12, 2015
(Prof. Permuter Haim and Mr. Oron Sabag)

Final Exam - Moed A 12.07.15
Total time for the exam: 3 hours!

Important: When **true** or **false** question appears, you should prove the statement if true, and provide counterexample otherwise.

1) **Bounds on entropy (15 Points)**:
Let X be a discrete random variable with $p_m = \max_{x\in\mathcal{X}} P(X = x)$.

 a) Is it true that

  i) $H(X) \leq H_b(p_m) + (1 - p_m)\log(|\mathcal{X}| - 1)$,
  ii) $H(X) \leq 1 + (1 - p_m)\log(|\mathcal{X}| - 1)$,
  where $H_b(x) = -x \log x - (1 - x)\log(1 - x)$ for $x \in [0, 1]$.

 Copy each relation to your notebook and write **true** or **false**.
 b) What are the conditions for which the equality $H(X) = H_b(p_m) + (1 - p_m)\log(|\mathcal{X}| - 1)$ holds?
 c) What are the conditions for which the equality $H(X) = 1 + (1 - p_m)\log(|\mathcal{X}| - 1)$ holds?

2) **True or False of a constrained inequality (21 Points)**:
Given are three discrete random variables X, Y, Z that satisfy $H(Y|X, Z) = 0$.

 a) Copy the next relation to your notebook and write **true** or **false**.

$$I(X; Y) \geq H(Y) - H(Z)$$

 b) What are the conditions for which the equality $I(X; Y) = H(Y) - H(Z)$ holds.
 c) Assume that the conditions for $I(X; Y) = H(Y) - H(Z)$ are satisfied. Is it true that there exists a function such that $Z = g(Y)$?

3) **Horstein scheme - Ternary erasure channel (32 Points)**
Given is the ternary erasure channel described in Fig. 1. The input alphabet is $\mathcal{X} = \{0, 1, 2\}$, while the output alphabet is $\{0, 1, 2, ?\}$. The probability for successful transmission, i.e. $P_{Y|X}(x|x) = 1 - \epsilon$, for all $x \in \mathcal{X}$, and the probability for erasure is ϵ, i.e. $P_{Y|X}(?|x) = \epsilon$, for all $x \in \mathcal{X}$.

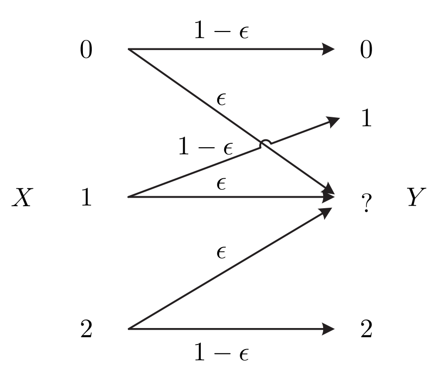

Fig. 1: Ternary erasure channel.

 a) What is the capacity of this channel?
 b) What is the optimal input distribution?
 c) Construct encoder and decoder based on the Horstein scheme for this channel. Specifically, write X_i as a function of θ and $p(m|Y^{i-1})$, where θ is the correct message. In addition, write the update of the posterior, i.e. write $p(m|Y^i)$ as a function of $p(m|Y^{i-1})$ and Y_i.
 You can use results from homework to simplify your solution with appropriate explanation.
 d) Given is sequence of outputs $(y_1, y_2, y_3, y_4) = (0, ?, 1, 0)$, draw the function $p(m|y^4)$ using the previous question.

4) **Choice of channels (32 Points)**
We define a union of 2 channels $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$ where, at each time, one can send a symbol over channel 1 **or** over channel 2 but not both. Assume the output alphabets are distinct and do not intersect. The capacity of this channel is defined as C, and the capacity of each channel $(\mathcal{X}_i, p_i(y_i|x_i), \mathcal{Y}_i)$ is defined as C_i for $i = 1, 2$.

a) Copy the next relation to your notebook and write **true** or **false**.

$$2^C = 2^{C_1} + 2^{C_2}.$$

b) What is the capacity of the channel described in Fig. 2, as a function of $\alpha \in [0, 1]$?

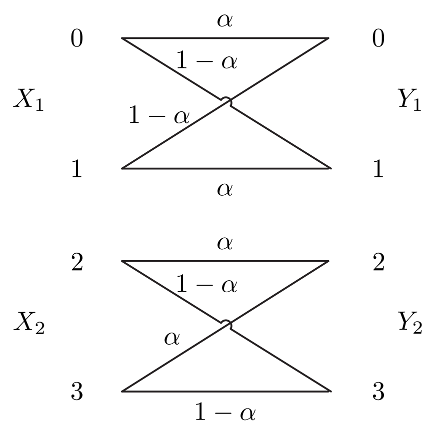

Fig. 2: Two binary channels.

Good Luck!