Multi-User Information Theory 2 January 3rd, 2013

Lecture 9

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I. BINARY BROADCAST CHANNEL

Definition 1 (Binary BC) We define the *Binary Broadcast Channel* depicted in Fig. 1. We set the following distributions

$$Z_1 \sim B(p_1) \tag{1}$$

$$Z_2 \sim B(p_2) \tag{2}$$

$$\tilde{Z}_2 \sim B(\tilde{p_2}) \tag{3}$$

$$Z_1 \perp Z_2 \tag{4}$$

$$Z_2 = Z_1 \oplus \tilde{Z}_2 \tag{5}$$

thus $p_2 = p_1 * \tilde{p_2} = p_1(1 - p_2) + (1 - p_1)p_2$. Additionally, we define

$$Y_1 = X \oplus Z_1 \tag{6}$$

$$Y_2 = Y_1 \oplus \tilde{Z}_2 = X \oplus Z_1 \oplus \tilde{Z}_2 = X \oplus Z_2 \tag{7}$$

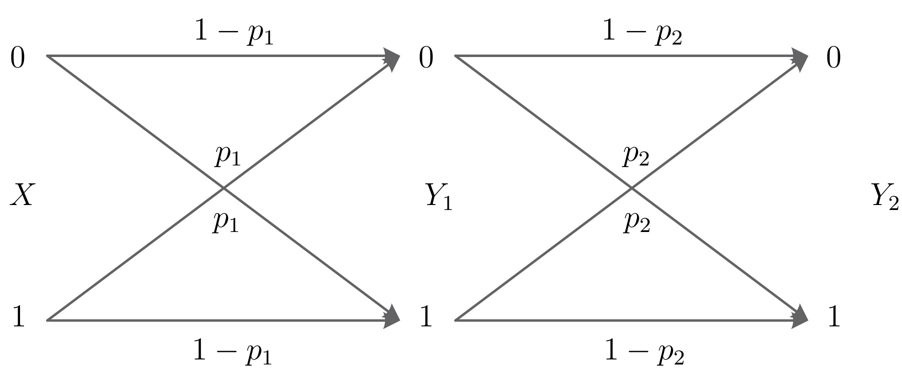

Fig. 1. The Binary Broadcast Channel

The capacity region of the degraded broadcast channel is the set of all pair rates (R_1, R_2) that satisfies

$$\begin{aligned} R_1 &< I(X;Y_1|U) \\ R_2 &< I(U;Y_2) \end{aligned} \tag{8}$$

for some joint distribution $p(u)p(x|u)p(y_1,y_2|x)$. We state the capacity of the Binary Broadcast Channel.

Lemma 1 (Capacity of the Binary BC) The capacity region of the Binary Broadcast Channel is the of all pairs (R_1, R_2) that satisfies

$$R_1 < h(\alpha * p_1) - h(p_1) \tag{9}$$

$$R_2 < 1 - h(\alpha * p_2). \tag{10}$$

Where $\alpha \in [0, 1]$.

Proof:

Achievability: Let us define two r.v $U \sim B(\frac{1}{2}), V \sim B(\alpha)$ and change $X = U \oplus V$. We now apply these new definitions to (8)

$$R_2 = I(U, Y_2) \tag{11}$$

$$= H(Y_2) - H(Y_2|U) \tag{12}$$

$$= 1 - h(\alpha * p_2) \tag{13}$$

(14)

Furthermore,

$$R_1 = I(X;Y_1|U) \tag{15}$$

$$= I(U \oplus V; U \oplus V \oplus Z_1|U) \tag{16}$$

$$= I(V; V \oplus Z_1) \tag{17}$$

$$= h(V \oplus Z_1) - h(V \oplus Z_1|V) \tag{18}$$

$$= h(\alpha \oplus p_1) - h(p_1) \tag{19}$$

Thus we obtained the desired region and proved the achievability part. In order to prove the converse we introduce the following lemma.

Lemma 2 (Mrs. Gerber Lemma) Let X and U be two random variables where X is binary. If $Y \sim B(P)$ is independent of (X, U) then

$$h(X \oplus Y|U) \geq h(h^{-1}(h(X|U)) * p) \tag{20}$$

where $a * b = \bar{a}b + b\bar{a}$ and h is the binary entropy.

The similarity between the Mrs Gerber Lemma and the EPI is shown in the Appendix. We now return to the converse,

Converse:

$$\begin{aligned} I(Y_2; U) &= h(Y_2) - h(Y_2|U) && (21) \\ &\leq 1 - h(Y_2|U) && (22) \end{aligned}$$

We bound $h(Y_2|U)$ from both sides

$$1 = h(Y_2|U) \geq h(Y_2|X) = h(p_2) \tag{23}$$

by the Markov chain $Y_2 - X - U$. Therefore there exists an α s.t

$$h(Y_2|U) = h(\alpha \oplus p_2) \tag{24}$$

thus

$$R_2 < 1 - h(\alpha \oplus p_2) \tag{25}$$

Now for R_1,

$$\begin{aligned} I(X; Y_2|U) &= h(Y_1|U) - h(Y_1|U, X) && (26) \\ &= h(Y_1|U) - h(p_1) && (27) \end{aligned}$$

By setting $X = Y_1, Y = Z_2$ in the Mrs. Gerber Lemma we obtain

$$\begin{aligned} h(Y_2|U) &\geq h(h^{-1}(h(Y_1|U)) * \tilde{p_2}) && (28) \\ h(\alpha * p_2) &\geq h(h^{-1}(h(Y_1|U)) * \tilde{p_2}) && (29) \\ \alpha * p_1 * \tilde{p_2} &\geq h^{-1}(h(Y_1|U)) * \tilde{p_2} && (30) \end{aligned}$$

$$\alpha * p_1 \geq h^{-1}(h(Y_1|U)) \quad (31)$$

$$h(\alpha * p_1) \geq h(Y_1|U))) \quad (32)$$

therefore,

$$R_1 \leq h(\alpha * p_1) - h(p_1) \quad (33)$$

which concludes the proof. ■

II. APPENDIX

A. A binary analogue to the EPI

We will introduce an analogue between the EPI and the Mrs. Gerber Lemma. The EPI states that for any independent $X \sim f(x)$ and $Z \sim f(z)$

$$2^{2h(X+Y)} \geq 2^{2h(X)} + 2^{2h(Y)} \quad (34)$$

If we define $\eta(x) = \frac{1}{2}\log(2\pi e x)$ then,

$$\eta^{-1}(h(X+Y)) \geq \eta^{-1}(h(X)) + \eta^{-1}(h(Y)) \quad (35)$$

Now let X and Y be independent r.v with finite alphabet and entropy $H(X) < 1, H(Y) < 1$ respectively. We define $\sigma(x) = h^{-1}(H(X))$ thus

$$\sigma(X \oplus Y) \geq \sigma(X) * \sigma(Y) \quad (36)$$

$$h^{-1}(H(X \oplus Y)) \geq h^{-1}(H(X)) * h^{-1}(H(Y)) \quad (37)$$

Notice the similarity of (35) and (37). Now we set $Y \sim B(p) \perp (X, U)$ thus we obtain

$$h^{-1}(H(X \oplus Y|U)) \geq h^{-1}(H(X|U)) * h^{-1}(H(Y|U)) \quad (38)$$

$$h^{-1}(H(X \oplus Y|U)) \geq h^{-1}(H(X|U)) * p \quad (39)$$

taking $h(\cdot)$ on both sides (which is possible since h is monotonically increasing), we obtain the Mrs. Gerber Lemma.

REFERENCES

[1] Abbas El Gamal and Young-Han Kim, ”*Network Information Theory*”, Lecture notes, Available online
[2] Thomas Cover and Joy Thomas, ”*Elements of Information Theory*”, 2nd ed., Wiley- Interscience, 2006.