Multi-User Information Theory 2 December 13th, 2012

Lecture 6

Lecturer: Haim Permuter *Scribe: Tal Kopetz*

I. Gaussian Broadcast Channel

Let us consider the Gaussian Broadcast Channel depicted in Fig. 1 where for $i \in [1,2]$, $Z_i \sim N(0,\sigma_i^2)$ and $Z_1 \perp Z_2$. Additionaly, there is a power constraint on the input $E[\sum_{i=1}^n X_i^2] \leq P$.

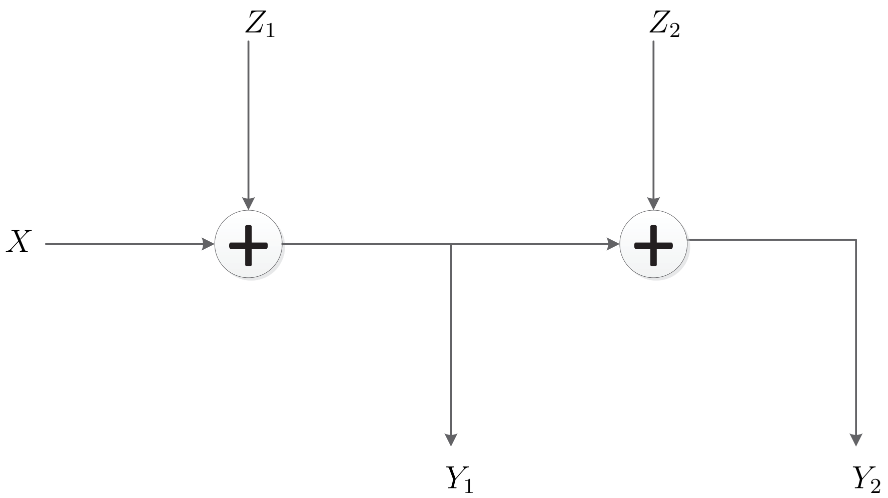

Fig. 1. The Gaussian Broadcast Channel

The capacity of the degraded broadcast channel is the set of all pairs (R_1,R_2) that satisfies

$$R_1 < I(X;Y_1|U) \tag{1}$$

$$R_2 < I(U;Y_2) \tag{2}$$

for some joint distribution $p(u)p(x|u)p(y_1,y_2|x)$.

Theorem 1 (Gaussian BC) The capacity region of the gaussian broadcast channel is the set of all pairs (R_1, R_2) that satisfies

$$R_1 < \frac{1}{2}\log(1+\frac{\alpha P}{\sigma_1^2}) \quad (3)$$

$$R_2 < \frac{1}{2}\log(1+\frac{\bar{\alpha} P}{\alpha P+\sigma_1^2+\sigma_2^2}) \quad (4)$$

Where $\alpha \in [0,1]$.

Proof: We will find the capacity region of the gaussian broadcast channel under the power constraint of $\frac{1}{n}\sum_{i=1}^{n} E[X_i^2] \leq P$.

Proof of Achievability: We start by setting the following distributions:

$$Z_i \sim N(0,\sigma_i^2) \quad (5)$$

$$X \sim N(0,P) \quad (6)$$

$$U \sim N(0,\alpha P) \quad (7)$$

$$V \sim N(0,\bar{\alpha} P) \quad (8)$$

where $0 \leq \alpha \leq 1$ and $\bar{\alpha} = 1-\alpha$. Therefore

$$R_1 = I(X;Y_1|U) \quad (9)$$

$$= I(U+V;U+V+Z_1|U) \quad (10)$$

$$= I(V;V+Z_1) \quad (11)$$

$$= \frac{1}{2}\log(1+\frac{\alpha P}{\sigma_1^2}) \quad (12)$$

and similarly,

$$R_2 = I(U;Y_2) \quad (13)$$

$$= I(U;U+V+Z_1+Z_2) \quad (14)$$

$$= \frac{1}{2}\log(1+\frac{\bar{\alpha} P}{\alpha P+\sigma_1^2+\sigma_2^2}) \quad (15)$$

thus obtaining an achievable region. ■

In order to prove the converse we will use the following lemma.

Lemma 1 (Entropy power inequality) The *Entropy Power Inequality* (EPI) states that for any independent $X \sim f(x)$ and $Z \sim f(z)$

$$2^{2h(X+Z)} \geq 2^{2h(X)} + 2^{2h(Z)} \tag{16}$$

in the vector case, where $X^n \sim f(x^n)$, $Z^n \sim f(z^n)$,

$$2^{\frac{2}{n}h(X^n+Z^n)} \geq 2^{\frac{2}{n}h(X^n)} + 2^{\frac{2}{n}h(Z^n)} \tag{17}$$

and in the conditional case

$$2^{2h(X+Z|U)} \geq 2^{2h(X|U)} + 2^{2h(Z|U)} \tag{18}$$

Lemma 2 (Alternative representation of EPI) Let X, Z be independent r.v and X', Z' gaussian independent r.v. If $h(Z) = h(Z')$ and $h(X) = h(X')$ then

$$2^{2h(X+Z)} \quad \geq \quad 2^{2h(X'+Z')} \tag{19}$$

is equivalent to (16).

Proof:

$$\begin{aligned}
2^{2h(X+Z)} &\geq 2^{2h(X'+Z')} && (20)\\
&= 2^{2\frac{1}{2}\log(2\pi e(\sigma_x^2+\sigma_z^2))} && (21)\\
&= 2\pi e(\sigma_x^2+\sigma_z^2) && (22)\\
&= 2^{2\frac{1}{2}\log(2\pi e\sigma_x^2)} + 2^{2\frac{1}{2}\log(2\pi e(\sigma_z^2))} && (23)\\
&= 2^{2h(X')} + 2^{2h(Z')} && (24)\\
&= 2^{2h(X)} + 2^{2h(Z)} && (25)
\end{aligned}$$

Thus we have shown that (16) is equivalent to (19). ■

Proof for the EPI conditional case given the scalar case: We will now prove the conditional case (18) based on the scalar case (16). We need to show that

$$2^{2h(X+Z|U)} \geq 2^{2h(X|U)} + 2^{2h(Z|U)} \tag{26}$$

or equivalently,

$$2\sum_{u\in\mathcal{U}} p(u)h(X+Z|U=u) \geq \log(2^{2\sum_{u\in\mathcal{U}} p(u)h(X|U=u)} + 2^{2\sum_{u\in\mathcal{U}} p(u)h(Z|U=u)}) \tag{27}$$

Let us consider the following function

$$f(x,y) = \ln(e^x + e^y) \tag{28}$$

we will show that $f(x,y)$ is convex in the pair (x,y) by showing that its hessian is positive semi-definite.

$$\frac{\partial^2 f(x,z)}{\partial x^2} = \frac{\partial^2 f(x,z)}{\partial z^2} = -\frac{\partial^2 f(x,z)}{\partial x \partial z} = -\frac{\partial^2 f(x,z)}{\partial y \partial x} = \frac{e^{x+z}}{(e^x + e^z)^2} \tag{29}$$

thus,

$$\begin{pmatrix} \frac{\partial^2 f(x,z)}{\partial x^2} & \frac{\partial^2 f(x,z)}{\partial x \partial z} \\ \frac{\partial^2 f(x,z)}{\partial z \partial x} & \frac{\partial^2 f(x,z)}{\partial z^2} \end{pmatrix} = \begin{pmatrix} \frac{e^{x+z}}{(e^x+e^z)^2} & -\frac{e^{x+z}}{(e^x+e^z)^2} \\ -\frac{e^{x+z}}{(e^x+e^z)^2} & \frac{e^{x+z}}{(e^x+e^z)^2} \end{pmatrix} \tag{30}$$

$$= \frac{e^{x+z}}{(e^x + e^z)^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{31}$$

$$= \frac{e^{x+z}}{(e^x + e^z)^2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{32}$$

which means that the hessian is positive semi-definite thus $f(x,z)$ is convex for any pair (x,z). Now we set $x = h(X|U=u), z = h(Z|U=u)$. From the convexity of $f(x,z)$, by Jensen's Inequality,

$$\sum_{u \in \mathcal{U}} p(u) \ln(e^{h(X|U=u)} + e^{h(Z|U=u)} \geq \ln(e^{\sum_{u\in\mathcal{U}} p(u) h(X|U=u)} + e^{\sum_{u\in\mathcal{U}} p(u) h(Z|U=u)}) \tag{33}$$

which is the same as in our problem hence (18) holds. ■

We now proceed with the converse.

Proof of Converse: By Fano's Inequality,

$$R_2 < I(Y_2;U) = h(Y_2) - h(Y_2|U) \tag{34}$$

For the first term

$$h(Y_2) \leq \frac{1}{2}\log(2\pi e(P + \sigma_1^2 + \sigma_2^2)), \tag{35}$$

where (a) follows from the concavity of log function. We now bound the second term as follows

$$\frac{1}{2}\log(2\pi e(\sigma_1^2 + \sigma_2^2)) = h(Y_2|X) \leq h(Y_2|U) \leq h(Y_2) \leq \frac{1}{2}\log(2\pi e(P + \sigma_1^2 + \sigma_2^2)) \tag{36}$$

by the Markov chain $Y_2 - X - U$. From the two bounds we conclude that there must exist some $0 \leq \alpha \leq 1$ s.t

$$h(Y_2|U) = \frac{1}{2}\log(2\pi e(\alpha P + \sigma_1^2 + \sigma_2^2)) \tag{37}$$

and by combining (35) and (38) we obtain

$$R_2 \leq \frac{1}{2}\log(1 + \frac{\bar{\alpha}}{\alpha P + \sigma_1^2 + \sigma_2^2}) \tag{38}$$

We now continue to R_1. By Fano's Inequality,

$$\begin{aligned}
R_1 &\leq I(X;Y_1|U) && (39)\\
&= h(Y_1|U) - h(Y_1|X,U) && (40)\\
&= h(Y_1|U) - h(Y_1|X) && (41)\\
&\overset{(a)}{\leq} \frac{1}{2}\log(2\pi e(\alpha P + \sigma_1^2)) - \frac{1}{2}\log(2\pi e\sigma_1^2) && (42)\\
&= \frac{1}{2}\log(1 + \frac{\alpha P}{\sigma_1^2}) && (43)
\end{aligned}$$

where (a) follows from the EPI since

$$2^{2h(Y_2|U)} \geq 2^{2h(Y_1|U)} + 2^{2h(Z_2|U)} \tag{44}$$

therefore,

$$\begin{aligned}
2^{2h(Y_1|U)} &\leq 2^{2h(Y_2|U)} - 2^{2h(Z_2|U)} && (45)\\
&= 2\pi e(\alpha P + \sigma_1^2 + \sigma_2^2) - 2\pi e\sigma_2^2 && (46)\\
&= 2\pi e(\alpha P + \sigma_1^2) && (47)
\end{aligned}$$

thus we conclude that

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log(1 + \frac{\alpha P}{\sigma_1^2}) && (48)\\
R_2 &\leq \frac{1}{2}\log(1 + \frac{\bar{\alpha}}{\alpha P + \sigma_1^2 + \sigma_2^2}) && (49)
\end{aligned}$$

■

Bergmans (1974) established the converse for the capacity region of the Gaussian BC using the entropy power inequality. The EPI was first stated by Shannon (1948) in [4]. the first formal proofs are due to Stam [5] and Blachman [6]. More versions of the EPI are available in [1] and [2]. For further reading, see references below.

II. APPENDIX

A. The duality between the EPI and the Brunn-Minkowski Inequality

We introduce the following theorem from mathematics.

Theorem 2 (Brunn-Minkowski Inequality) The volume of the set-sum of two sets A and B is greater than the volume of the set-sum of two spheres A', B' with the same volume as A' and B'. In other words

$$Vol(A+B) \geq Vol(A'+B') \tag{50}$$

$\forall A', B'$ s.t $Vol(A) = Vol(A')$ and $Vol(B) = Vol(B')$.

The Brunn-Minkowski Inequality (BMI) is very similar to the EPI. In information theory, the differential entropy $h(X)$ relates to volume in the following way:

Let $\{X_i\}_{i\geq 1}$ be an i.i.d process with a probability density function $f(x)$. Also, let S_n be a sequence of sets s.t

$$\lim_{n\to\infty} \Pr(x^n \in S_n) = 1 \tag{51}$$

Then

$$\limsup_{n\to\infty} Vol(S_n) \geq 2^{nh(x)} \tag{52}$$

and for any $\epsilon > 0$ there exists a sequence of volumes s.t

$$\lim_{n\to\infty} Vol(S_n) \leq 2^{n(h(x)+\epsilon)} \tag{53}$$

Hence, we can see that the volume of the set-sum in the BMI is the analogue of $h(X+Z)$ in the EPI.

REFERENCES

[1] Abbas El Gamal and Young-Han Kim, *"Network Information Theory"*, Lecture notes, Available online

[2] Thomas Cover and Joy Thomas, *"Elements of Information Theory"*, 2nd ed., Wiley- Interscience, 2006.

[3] Patrick P. Bergmans, *"A simple converse for broadcast channels with additive white Gaussian noise,* IEEE Trans. Inform. Theory, vol. IT-20, pp. 279280.

[4] Claude E. Shannon, *"A mathematical theory of communication"*. Bell System Tech. J. 27: 379423, 623656. 1948

[5] A. Stam. *"Some inequalities satisfied by the quantities of information of Fisher and Shannon"*. Inf. Control, 2:101112, June 1959.

[6] N. Blachman. *"The convolution inequality for entropy powers"*. IEEE Trans. Inf. Theory, IT-11:267271, Apr. 1965.