Multi-User Information Theory 2. December 6nd, 2010

Lecture 9

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I. Stochastic Degraded Broadcast Channel

Recall, the definition of the physically degraded broadcast channel (BC).

Definition 1 (Physically Degraded BC)
A broadcast channel given by $p(y_1, y_2|x)$ is said to be *(physically degraded)* if

$$p(y_1, y_2|x) = p(y_1|x)p(y_2|y_1),\ \forall x \in \mathcal{X},\ \forall y_1 \in \mathcal{Y}_1,\ \forall y_2 \in \mathcal{Y}_2. \tag{1}$$

Let us define the stochastic degraded BC,

Definition 2 (Stochastic degraded BC)
A broadcast channel given by $p(y_1, y_2|x)$ is said to be *(stochastic degraded)* if its conditional marginal distributions $p(y_1|x)$ and $p(y_2|x)$ are the same as that of a physically degraded broadcast channel; that is, if there exists a distribution $p'(y_2|y_1)$ such that

$$p(y_2|x) = \sum_{y_1} p(y_1|x)p'(y_2|y_1). \tag{2}$$

Theorem 1 : The capacity region of a broadcast channel depends only on the conditional marginal distributions $p(y_1|x)$ and $p(y_2|x)$.

Proof: See HW 2. Hint: for any given $(2^{nR1}, 2^{nR2}, n)$ code, let

$$\begin{aligned} P_1^{(n)} &= P\{\hat{W}_1(Y_1) \neq W_1\}, \\ P_2^{(n)} &= P\{\hat{W}_2(Y_2) \neq W_2\}, \\ P^{(n)} &= P\{\hat{W}_1, \hat{W}_2 \neq W_1, W_2\}. \end{aligned}$$

Then show

$$max\{P_1^{(n)}, P_2^{(n)}\} \leq P^{(n)} \leq P_1^{(n)} + P_2^{(n)}.$$

■

Note that since the capacity of a BC depends only on the conditional marginals, the capacity region of the stochastically degraded BC is the same as that of the corresponding physically degraded channel.

Example 1 (Gaussian BC is stochastic degraded)
The Gaussian BC is illustrated in Fig. 1. It can easily be shown that all scalar Gaussian broadcast channels

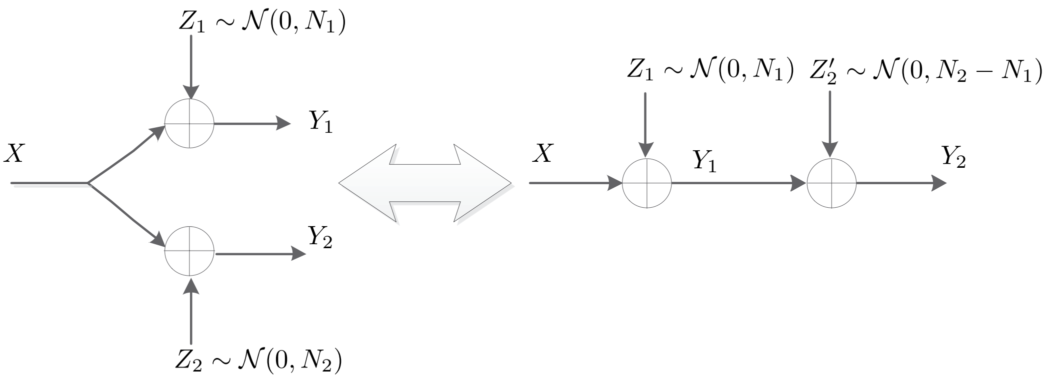

Fig. 1. Gaussian BC.

are equivalent to this type of degraded channel,

$$\begin{aligned} Y_1 &= X + Z_1, \\ Y_2 &= X + Z_2 = Y_1 + Z_2', \end{aligned}$$

where $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z_2 \sim \mathcal{N}(0, N_2 - N_1)$.

Example 2 (Capacity region of binary symmetric BC)

Consider a pair of binary symmetric channels with parameters p_1 and p_2 that form a BC as shown in Fig. 2. Without loss of generality, we assume that $p_1 \leq p_2 \leq \frac{1}{2}$. This is stochastic degraded channel since we

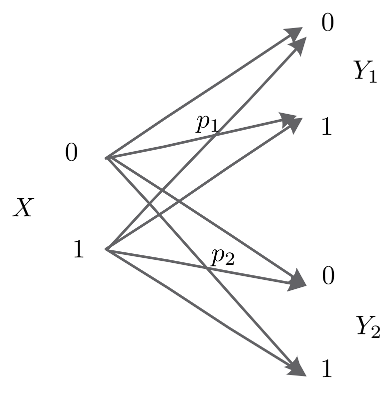

Fig. 2. Binary symmetric BC.

can recast this channel as a physically degraded channel. Namely, we can express the binary symmetric channel with parameter p_2 as a cascade of a binary symmetric channel with parameter p_1 with another binary symmetric channel. Let the crossover probability of the new channel be α. Then we must have

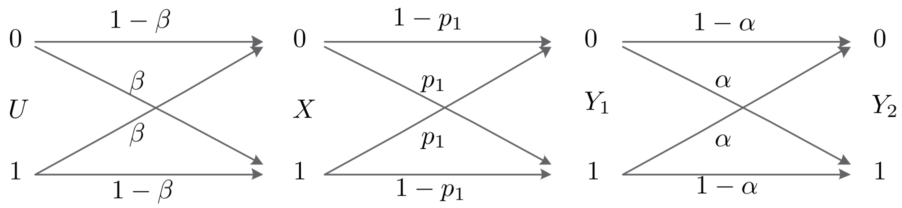

Fig. 3. Stochastic degraded binary symmetric BC.

$$p_1(1-\alpha)+(1-p_1)\alpha \quad = \quad p_2, \tag{3}$$

therefore,

$$\alpha \quad = \quad \frac{p_2-p_1}{1-2p_1},$$

Note that under the assumption of $p_1 \leq p_2 \leq \frac{1}{2}$, $0 \leq \alpha \leq \frac{1}{2}$. Now, by symmetry we connect U to X by another binary symmetric channel with parameter β, as illustrated in Fig. 3. Let us calculate the rates in the capacity region. It is clear by symmetry, that the distribution on U that maximizes the rates is the $U \sim Binary(0.5)$, Therefore,

$$\begin{aligned} I(U;Y_2) &= H(Y_2)-H(Y_2|U) \\ &= 1-H(\beta * p_2), \end{aligned}$$

where

$$\beta * p_2 \quad = \quad \beta(1-p_2)+(1-\beta)p_2.$$

Similarly,

$$\begin{aligned} I(X;Y_1|U) &= H(Y_1|U)-H(Y_1|U,X) \\ &= H(Y_1|U)-H(Y_1|X) \\ &= H(\beta * p_1)-H(p_1). \end{aligned}$$

Hence, the capacity region of a binary symmetric BC is:

$$\begin{aligned} R_1 &\leq H(\beta * p_1)-H(p_1), \\ R_2 &\leq 1-H(\beta * p_2), \end{aligned}$$

(4)

for some $\beta \in [0, \frac{1}{2}]$. When $\beta = 0$, we have the maximum information transfer to Y_2, i.e., ($R_1 = 0, R_2 = 1 - H(p_2)$). When $\beta = 0.5$, we have the maximum information transfer to Y_1, i.e., ($R_1 = 1 - H(P_1), R_2 = 0$). For an additional proof of the capacity region of a binary symmetric broadcast channel see [1, Chapter 5].

II. Inner Bound for General Broadcast Channel- Marton Region

In this section we prove the inner bound (achievability) for the general broadcast channel [2], which is the best achievable region known for the general broadcast channel.

To prove the achievability of the capacity region, we need to show that for a fix $p(u_1, u_2)P(x|u_1, u_2)P(y_1, y_2|x)$ and (R_1, R_2) that satisfy

$$\begin{aligned} R_1 &\leq I(U_1; Y_1), \\ R_2 &\leq I(U_2; Y_2), \\ R_1 + R_2 &\leq I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2), \end{aligned}$$

there exists a sequence of $(n, 2^{nR_1}, 2^{nR_2})$ codes where $P_e^{(n)} \to 0$ as $n \to \infty$.

Proof: We prove the inner bound by achieving the two corner points of the rate region, i.e., ($R_1 = I(U_1; Y_1) - \epsilon, R_2 = I(U_2; Y_2) - I(U_1; U_2) - \epsilon$), and ($R_1 = I(U_1; Y_1) - I(U_1; U_2) - \epsilon, R_2 = I(U_2; Y_2) - \epsilon$).

Code design:

1) Generate 2^{nR_1} independent codewords of length n, U_1^n, according to $U_1^n(m_1) \sim$ i.i.d. $p(u_1)$. To each codeword U_1^n associate a message $m_1 \in \mathcal{M}_1 = \{1, 2, ..., 2^{nR_1}\}$.
2) Generate 2^{nR_2} bins, in each bin generate $2^{n(I(U_1;U_2)+\epsilon)}$ independent codewords of length n, U_2^n, according to $U_2^n \sim$ i.i.d. $p(u_2)$. To each bin associate a message $m_2 \in \mathcal{M}_2 = \{1, 2, ..., 2^{nR_2}\}$.

Encoder: Given a pair of messages (m_1, m_2), the encoder first choose the U_1^n that associated with m_1, i.e., $U_1^n(m_1)$. Then, search for U_2^n in bin m_2, such that

$$(U_1^n(m_1), U_2^n) \in T_\epsilon^n(U_1, U_2). \tag{5}$$

If the encoder found such U_2^n in bin m_2, it transmits X^n, where $X^n = f(U_1^n, U_2^n)$. Otherwise, the encoder declares an error.

Decoder 1: Search for U_1^n such that

$$(U_1^n, Y_1^n) \in T_\epsilon^n(U_1, Y_1). \tag{6}$$

If decoder 1 found such U_1^n, it declares the message $\hat{m}_1$ that associated with U_1^n. Otherwise, decoder 1 declares an error.

Decoder 2: Search for U_2^n such that

$$(U_2^n, Y_2^n) \in T_\epsilon^n(U_2, Y_2). \tag{7}$$

If decoder 2 found such U_2^n, it declares the bin that contains U_2^n to be the message $\hat{m}_2$. Otherwise, decoder 2 declares an error.

Analysis of the probability of error: Without loss of generality, we assume that $(M_1 = 1, M_2 = 1)$ was sent. Furthermore, we denote by $U_2^n(k_1, k_2)$ to be the k_1 codeword in the bin k_2, where $k_1 \in \mathcal{K}_1 = \{1, 2, .., 2^{nI(U_1,U_2)+\epsilon}\}$, $k_2 \in \mathcal{K}_2 = \{1, 2, .., 2^{nR_2}\}$. Let us define the following events:

$$
\begin{aligned}
E_1 &= \{(U_1^n(1), U_2^n(k_1,1)) \notin T_\epsilon^n(U_1,U_2),\ \forall k_1 \in \mathcal{K}_1\}. &(8)\\
E_2 &= \{(U_1^n(1), Y_1^n) \notin T_\epsilon^n(U_1,Y_1\}). &(9)\\
E_3 &= \{\exists i \neq 1 : (U_1^n(i), Y_1^n) \in T_\epsilon^n(U_1,Y_1),\ i \in \mathcal{M}_1\}. &(10)\\
E_4 &= \{(U_2^n(1,1), Y_2^n) \notin T_\epsilon^n(U_2,Y_1)\}. &(11)\\
E_5 &= \{\exists (k_1,k_2) \neq (1,1) : (U_2^n(k_1,k_2), Y_2^n) \in T_\epsilon^n(U_2,Y_2),\ (k_1,k_2) \in \mathcal{K}_1 \times \mathcal{K}_2\}. &(12)
\end{aligned}
$$

Then by the union of events bound,

$$
\begin{aligned}
P_e^n &= \Pr(\bigcup_{i=1}^{5} E_i) \\
&\leq \sum_{i=1}^{5} P(E_i). &(13)
\end{aligned}
$$

Now let us find the probability of each event,

- $P(E_1)$- in each bin, and in particular in bin 1, there is $2^{n(I(U_1;U_2)+\epsilon)}$ codewords U_2^n. Therefore, by using the covering lemma, as $n \to \infty$

$$P(E_1) \to 0. \tag{14}$$

- $P(E_2)$- using the L.L.N., as $n \to \infty$ the probability of error,

$$P(E_2) \to 0. \tag{15}$$

- $P(E_3)$- by the union of events bound, the probability of the event E_3,

$$
\begin{aligned}
P(E_3) &= \Pr\{\exists i \neq 1 : (U_1^n(i), Y_1^n) \in T_\epsilon^n(U_1,Y_1),\ i \in \mathcal{M}_1\} \\
&\leq \sum_{i=2}^{2^{nR_1}} P((U_1^n(i), Y_1^n) \in T_\epsilon^n(U_1,Y_1)) \\
&\leq 2^{nR_1} \cdot 2^{-n(I(U_1,Y_1)-\epsilon)}. &(16)
\end{aligned}
$$

  For $P(E_3) \to 0$ as $n \to \infty$, we need to choose,

$$R_1 < I(U_1,Y_1) - \epsilon. \tag{17}$$

- $P(E_4)$- using the L.L.N., as $n \to \infty$ the probability of the event E_4,

$$P(E_4) \to 0. \tag{18}$$

- $P\left(E_5\right)$- by the union of events bound, the probability of the event E_5,

$$\begin{aligned}
P\left(E_5\right) &= \Pr\left\{\exists(k_1,k_2)\neq(1,1):(U_2^n(k_1,k_2),Y_2^n)\in T_\epsilon^n(U_2,Y_2),\ (k_1,k_2)\in\mathcal{K}_1\times\mathcal{K}_2\right\} \\
&\leq \sum_{(k_1,k_2)\neq(1,1)} P((U_2^n(k_1,k_2),Y_2^n)\in T_\epsilon^n(U_2,Y_2)) \\
&\leq 2^{n(I(U_1;U_2)+\epsilon)}\cdot 2^{nR_2}\cdot 2^{-n(I(U_2;Y_2)-\epsilon)}. \qquad (19)
\end{aligned}$$

For $P\left(E_5\right)\to 0$ as $n\to\infty$, we need to choose,

$$R_2 < I(U_2,Y_2) - I(U_1,U_2) - 2\epsilon. \qquad (20)$$

Thus, the total average probability of decoding error $P_e^{(n)}\to 0$ as $n\to\infty$ if $(R_1 = I(U_1;Y_1)-\epsilon, R_2 = I(U_2;Y_2)-I(U_1;U_2)-\epsilon)$. The achievability of the other corner point follows by changing the code design order. To show achievability of other points in $\mathcal{R}$, we use time sharing between corner points and points on the axes.

Thus, the probability of error, conditioned on a particular codeword being sent, goes to zero if the conditions of the following are met:

$$\begin{aligned}
R_1 &\leq I(U_1;Y_1), \\
R_2 &\leq I(U_2;Y_2), \\
R_1+R_2 &\leq I(U_1;Y_1)+I(U_2;Y_2)-I(U_1;U_2).
\end{aligned}$$

The above bound shows that the average probability of error, which by symmetry is equal to the probability for an individual pair of codewords (m_1,m_2), averaged over all choices of codebooks in the random code construction, is arbitrarily small. Hence, there exists at least one code $(n,2^{nR_1},2^{nR_2})$ with arbitrarily small probability of error. To complete the proof we use time-sharing to allow any (R_1,R_2) in the convex hull to be achieved. ■

REFERENCES

[1] A. El Gamal and Y. H. Kim. Lecture notes on network information theory. *CoRR*, abs/1001.3404, 2010.

[2] K. Marton. Acoding theorem for the discrete memory less broadcast channel. *IEEE Trans. Inf. Theor.*, 25:306–311, May 1979.