# Analogy Between Gambling and Measurement-Based Work Extraction

Dror Vinkler [1] Haim Permuter [1] Neri Merhav [2]

[1]Ben Gurion University

[2]Technion - Israel Institute of Technology

ISIT 2014

# Outline

- Past results and brief physics background
- The analogy
- Gambling on continuous random variables
- Consequences
  - Universal engine
  - Memory
- Summary and future work

# Past Results - Horse Race Gambling

## Kelly 1956

- A race of $m$ horses.

# Past Results - Horse Race Gambling

Kelly 1956

- A race of $m$ horses.
- $X_i$ - winning horse. $Y_i$ - side information.

# Past Results - Horse Race Gambling

Kelly 1956

- A race of $m$ horses.
- $X_i$ - winning horse. $Y_i$ - side information.
- $X_i, Y_i$ are pairwise i.i.d.

# Past Results - Horse Race Gambling

Kelly 1956

- A race of $m$ horses.
- $X_i$ - winning horse. $Y_i$ - side information.
- $X_i, Y_i$ are pairwise i.i.d.
- $S_n$ - gambler's capital after $n$ rounds,

$$S_1 = b_{X|Y}(X_1|Y_1)o_X(X_1)S_0,$$

$b_{X|Y}(X|Y)$ - betting strategy.
$o_X(X)$ - odds.

# Past Results - Horse Race Gambling

Kelly 1956

- A race of $m$ horses.
- $X_i$ - winning horse. $Y_i$ - side information.
- $X_i, Y_i$ are pairwise i.i.d.
- $S_n$ - gambler's capital after $n$ rounds,

$$S_1 = b_{X|Y}(X_1|Y_1)o_X(X_1)S_0,$$

$b_{X|Y}(X|Y)$ - betting strategy.
$o_X(X)$ - odds.

$$S_n = \prod_{i=1}^{n} b_{X|Y}(X_i|Y_i)o_X(X_i)S_0,$$

# Past Results - Horse Race Gambling

## Kelly 1956

- The goal: finding the optimal betting strategy $b^*_{X|Y}$,

$$b^*_{X|Y} = \arg\max_{b_{X|Y}} E\left[\log S_n\right].$$

$\log S_n$ - capital growth.

# Past Results - Horse Race Gambling

Kelly 1956

- The goal: finding the optimal betting strategy $b^*_{X|Y}$,

$$b^*_{X|Y} = \arg\max_{b_{X|Y}} E\left[\log S_n\right].$$

  $\log S_n$ - capital growth.

- This optimal betting strategy is $b^*_{X|Y} = P_{X|Y}$.

# Past Results - Horse Race Gambling

Kelly 1956

- The goal: finding the optimal betting strategy $b^*_{X|Y}$,

$$b^*_{X|Y} = \arg\max_{b_{X|Y}} E\left[\log S_n\right].$$

  $\log S_n$ - capital growth.

- This optimal betting strategy is $b^*_{X|Y} = P_{X|Y}$.
- A bet is *fair* if $o_X(x) = 1/P_X(x)\ \forall x$.

# Past Results - Horse Race Gambling

Kelly 1956

- The goal: finding the optimal betting strategy $b^*_{X|Y}$,

$$b^*_{X|Y} = \arg\max_{b_{X|Y}} E\left[\log S_n\right].$$

  $\log S_n$ - capital growth.

- This optimal betting strategy is $b^*_{X|Y} = P_{X|Y}$.

- A bet is *fair* if $o_X(x) = 1/P_X(x)\ \forall x$.

- For a fair bet, the maximal growth is $nI(X;Y)$.

# Brief Background on Statistical Mechanics

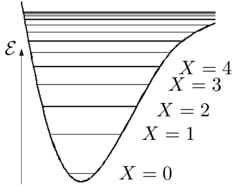
- For large systems, classical analysis is not possible.

# Brief Background on Statistical Mechanics

- For large systems, classical analysis is not possible.
- **Principle 1:**

# Brief Background on Statistical Mechanics

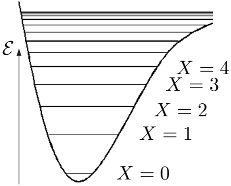
- For large systems, classical analysis is not possible.
- **Principle 1:**
  - X - state of particle.



# Brief Background on Statistical Mechanics

- For large systems, classical analysis is not possible.
- **Principle 1:**
  - X - state of particle.
  - For a fixed temperature,

$$P_X(x) \sim e^{-\mathcal{E}(x)/k_B T}.$$

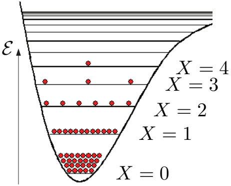



$P_X(x)$ - the Boltzmann distribution.

# Brief Background on Statistical Mechanics

- For large systems, classical analysis is not possible.
- **Principle 1:**
  - X - state of particle.
  - For a fixed temperature,

$$P_X(x) \sim e^{-\mathcal{E}(x)/k_B T}.$$

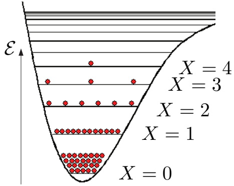



$P_X(x)$ - the Boltzmann distribution.

# Brief Background on Statistical Mechanics

- For large systems, classical analysis is not possible.
- **Principle 1:**
  - X - state of particle.
  - For a fixed temperature,

$$P_X(x) \sim e^{-\mathcal{E}(x)/k_B T}.$$



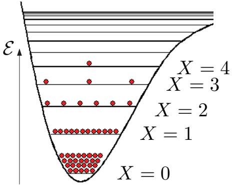

$P_X(x)$ - the Boltzmann distribution.

- **Principle 2:**

# Brief Background on Statistical Mechanics

- For large systems, classical analysis is not possible.
- **Principle 1:**
  - X - state of particle.
  - For a fixed temperature,

$$P_X(x) \sim e^{-\mathcal{E}(x)/k_B T}.$$




$P_X(x)$ - the Boltzmann distribution.

- **Principle 2:**
  - The second law of thermodynamics:

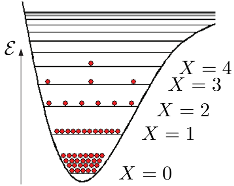
$$\Delta S \geq 0.$$

$S$ - the entropy of the system.

# Brief Background on Statistical Mechanics

- For large systems, classical analysis is not possible.
- **Principle 1:**
  - X - state of particle.
  - For a fixed temperature,

$$P_X(x) \sim e^{-\mathcal{E}(x)/k_B T}.$$




$P_X(x)$ - the Boltzmann distribution.

- **Principle 2:**
  - The second law of thermodynamics:

$$\Delta S \geq 0.$$

$S$ - the entropy of the system.

  - In a complete cycle: $E[W] \leq 0$.

# Past Results - Measurement-Based Work Extraction

- Originated with Maxwell's demon in the $19^{th}$ century.

# Past Results - Measurement-Based Work Extraction

- Originated with Maxwell's demon in the $19^{th}$ century.
- Using fluctuation theorems, it was shown that [Sagawa and Ueda 2010]

$$E[W] \leq k_B T I(X;Y).$$

# Past Results - Measurement-Based Work Extraction

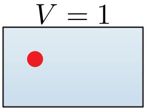
- Originated with Maxwell's demon in the 19$^{th}$ century.
- Using fluctuation theorems, it was shown that [Sagawa and Ueda 2010]

$$E[W] \leq k_B T I(X;Y).$$

- Some systems achieve this, but no general achievability scheme.

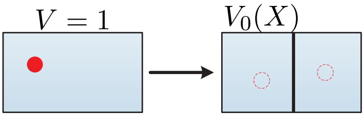
# Past Results - The Szilard Engine

- Presented by Sagawa and Ueda as an example:



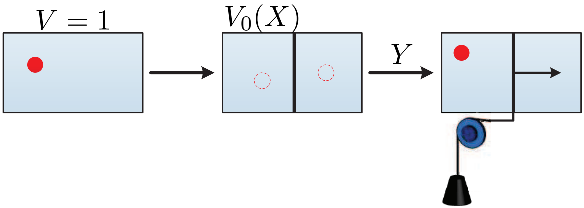
# Past Results - The Szilard Engine

- Presented by Sagawa and Ueda as an example:



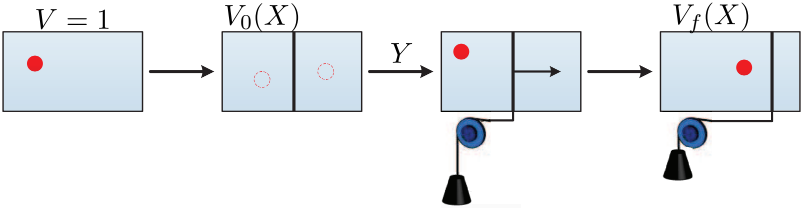
# Past Results - The Szilard Engine

- Presented by Sagawa and Ueda as an example:



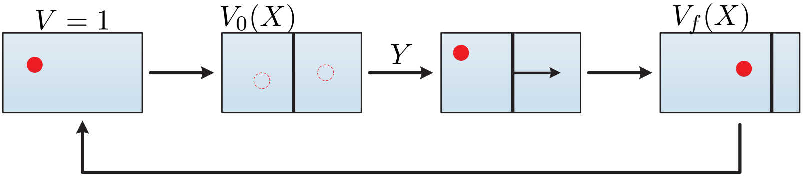
# Past Results - The Szilard Engine

- Presented by Sagawa and Ueda as an example:



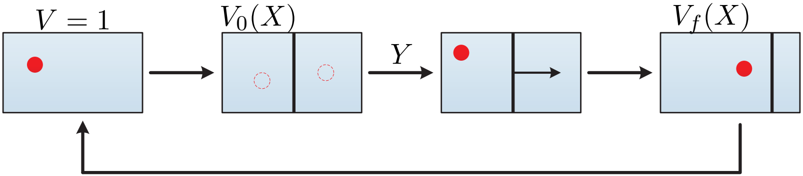
# Past Results - The Szilard Engine

- Presented by Sagawa and Ueda as an example:



# Past Results - The Szilard Engine

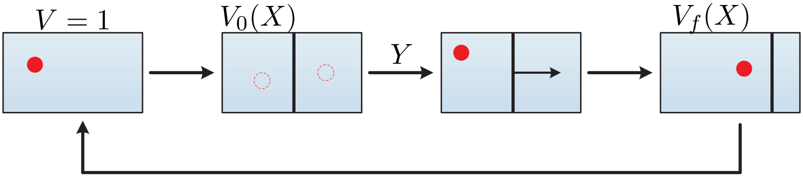
- Presented by Sagawa and Ueda as an example:




- $X$ - the particle's location. $Y$ - a noisy measurement.

# Past Results - The Szilard Engine

- Presented by Sagawa and Ueda as an example:




- $X$ - the particle's location. $Y$ - a noisy measurement.
- Extracted work is

$$W = k_B T \ln \frac{V_f(X|Y)}{V_0(X)}.$$

# Past Results - The Szilard Engine

- Optimal value of $V_f(X|Y)$ is

$$V_f^*(X|Y) = P_{X|Y}(X|Y).$$

# Past Results - The Szilard Engine

- Optimal value of $V_f(X|Y)$ is

$$V_f^*(X|Y) = P_{X|Y}(X|Y).$$

- Maximal extracted work is

$$\max E[W] = k_B T I(X;Y).$$

# Past Results - The Szilard Engine

- Optimal value of $V_f(X|Y)$ is

$$V_f^*(X|Y) = P_{X|Y}(X|Y).$$

- Maximal extracted work is

$$\max E[W] = k_B T I(X;Y).$$

- Upper bound is achieved.

# Past Results - The Szilard Engine

- Optimal value of $V_f(X|Y)$ is

$$V_f^*(X|Y) = P_{X|Y}(X|Y).$$

- Maximal extracted work is

$$\max E[W] = k_B T I(X;Y).$$

- Upper bound is achieved.
- Result is specific to one particle of ideal gas.

# Main Results

| **Gambling** | **Maxwell’s Demon** |
| --- | --- |
| Side information | Measurements results |

# Main Results

| **Gambling** | **Maxwell's Demon** |
|---|---|
| Side information | Measurements results |
| $o_X$ - odds | $1/V_0$ - initial vol. |

# Main Results

| Gambling | Maxwell's Demon |
|---|---|
| Side information | Measurements results |
| $o_X$ - odds | $1/V_0$ - initial vol. |
| $b_{X\|Y}$ - betting strategy | $V_f$ - final vol. |

# Main Results

| **Gambling** | **Maxwell's Demon** |
|---|---|
| Side information | Measurements results |
| $o_X$ - odds | $1/V_0$ - initial vol. |
| $b_{X\|Y}$ - betting strategy | $V_f$ - final vol. |
| $\log S_n$ - log of capital | $W_n$ - extracted work |

# Main Results

| **Gambling** | **Maxwell's Demon** |
| --- | --- |
| Side information | Measurements results |
| $o_X$ - odds | $1/V_0$ - initial vol. |
| $b_{X\|Y}$ - betting strategy | $V_f$ - final vol. |
| $\log S_n$ - log of capital | $W_n$ - extracted work |

$$\log S_n = \sum_{i=1}^{n} \log b_{X|Y}(X_i|Y_i) o_X(X_i) \ \ , \ \ W_n = \sum_{i=1}^{n} k_B T \ln \frac{V_f(X_i|Y_i)}{V_0(X_i)} .$$

# Some Immediate Consequences

- $o_X \Leftrightarrow 1/V_0 = 1/P_X$

# Some Immediate Consequences

- $o_X \Leftrightarrow 1/V_0 = 1/P_X \Rightarrow$ In physics, the bet is always fair.

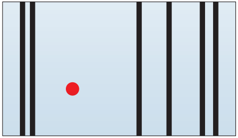
# Some Immediate Consequences

- $o_X \Leftrightarrow 1/V_0 = 1/P_X \Rightarrow$ In physics, the bet is always fair. Second law of thermo. - without measurements $E[W] \leq 0$.

# Some Immediate Consequences

- $o_X \Leftrightarrow 1/V_0 = 1/P_X \Rightarrow$ In physics, the bet is always fair. Second law of thermo. - without measurements $E[W] \leq 0$.
- Enables straightforward generalization to $m$ dividers

$$V_f^*(X|Y) = P_{X|Y}(X|Y).$$



# Recap

What we achieved so far:

# Recap

What we achieved so far:

- One-to-one mapping.

# Recap

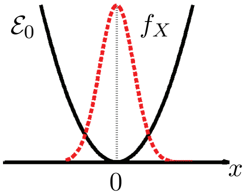
What we achieved so far:

- One-to-one mapping.
- Specific to Szilard engine.

# Continuous Random Variables - Physics

Horowitz and Parrondo 2011, Esposito and Van den Broeck 2011

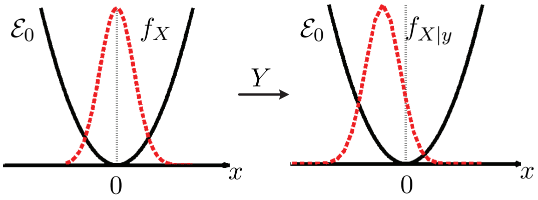
A single particle in a potential field:



# Continuous Random Variables - Physics

Horowitz and Parrondo 2011, Esposito and Van den Broeck 2011

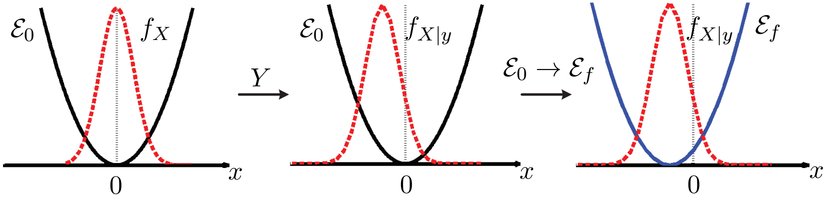
A single particle in a potential field:



# Continuous Random Variables - Physics

Horowitz and Parrondo 2011, Esposito and Van den Broeck 2011

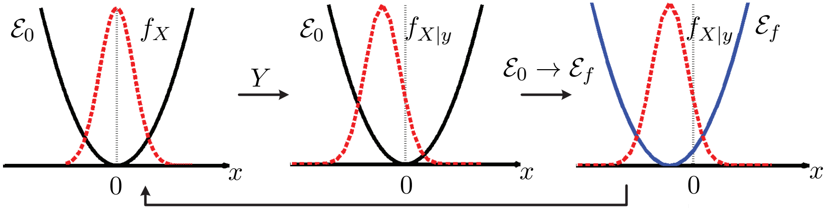
A single particle in a potential field:



# Continuous Random Variables - Physics

Horowitz and Parrondo 2011, Esposito and Van den Broeck 2011

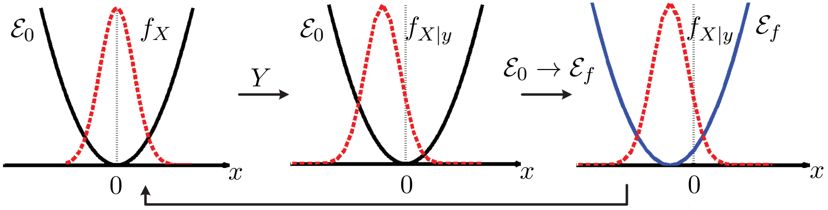
A single particle in a potential field:



# Continuous Random Variables - Physics

Horowitz and Parrondo 2011, Esposito and Van den Broeck 2011

A single particle in a potential field:

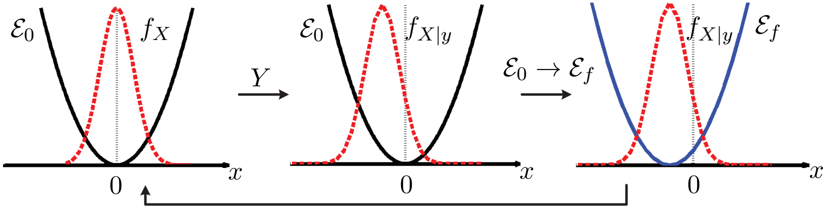



- $X$ - particle's location. $Y$ - noisy measurement.

# Continuous Random Variables - Physics

Horowitz and Parrondo 2011, Esposito and Van den Broeck 2011

A single particle in a potential field:

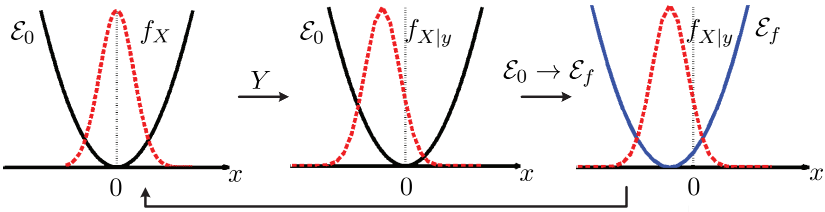



- $X$ - particle's location. $Y$ - noisy measurement.
- $Q_{X|y}$ - the Boltzmann distribution that stems from $\mathcal{E}_f$.

# Continuous Random Variables - Physics

Horowitz and Parrondo 2011, Esposito and Van den Broeck 2011

A single particle in a potential field:




- $X$ - particle's location. $Y$ - noisy measurement.
- $Q_{X|y}$ - the Boltzmann distribution that stems from $\mathcal{E}_f$.

**Theorem**

$$Q^*_{X|y} = \arg\min D(f_{X|y}||Q_{X|y}) \ \ \forall y \in \mathcal{Y}.$$

$$E[W] = k_B T \left[ I(X;Y) - D(f_{X|Y}||Q^*_{X|Y}|f_Y) \right].$$

# Continuous Random Variables - Gambling

What's the analogous case in gambling?

# Continuous Random Variables - Gambling

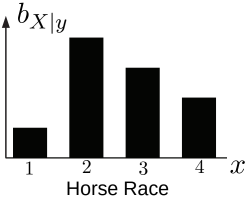
What's the analogous case in gambling?

- $X$ is some continuous r.v., i.e., the price of a stock.

# Continuous Random Variables - Gambling

What's the analogous case in gambling?

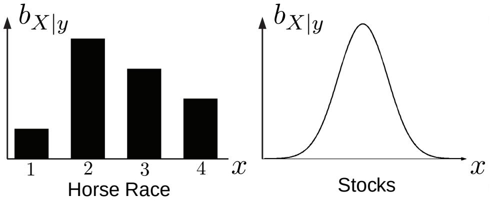
- $X$ is some continuous r.v., i.e., the price of a stock.
- Betting strategy:



# Continuous Random Variables - Gambling

What's the analogous case in gambling?

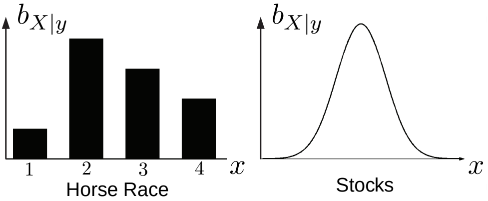
- $X$ is some continuous r.v., i.e., the price of a stock.
- Betting strategy:



# Continuous Random Variables - Gambling

What's the analogous case in gambling?

- $X$ is some continuous r.v., i.e., the price of a stock.
- Betting strategy:




**Theorem**

$$b^*_{X|y} = \arg\min D(f_{X|y} || b_{X|y}) \;\; \forall y \in \mathcal{Y}.$$

$$E[\log S^*_n] = n \left[ I(X;Y) - D(f_{X|Y} || b^*_{X|Y} | f_Y) \right].$$

# Consequence 1 - Universal Engine

<u>Gambling:</u>

- $X$, $Y$ discrete r.v. $P_{X,Y}$ is unknown.

# Consequence 1 - Universal Engine

Gambling:

- $X$, $Y$ discrete r.v. $P_{X,Y}$ is unknown.
- $\widehat{b}$ - the *universal portfolio* [Cover and Ordentlich 1996].

# Consequence 1 - Universal Engine

Gambling:

- $X$, $Y$ discrete r.v. $P_{X,Y}$ is unknown.
- $\widehat{b}$ - the *universal portfolio* [Cover and Ordentlich 1996].

**Theorem**

$$\lim_{n\to\infty}\frac{1}{n}E[\log\widehat{S}_n - \log S^*_n] = 0.$$

*$\widehat{S}_n$ - capital using $\widehat{b}$. $S^*_n$ - capital using $b^*_{X|Y}$.*

# Consequence 1 - Universal Engine

Gambling:

- $X$, $Y$ discrete r.v. $P_{X,Y}$ is unknown.
- $\widehat{b}$ - the *universal portfolio* [Cover and Ordentlich 1996].

**Theorem**

$$\lim_{n\to\infty} \frac{1}{n} E[\log \widehat{S}_n - \log S_n^*] = 0.$$

$\widehat{S}_n$ *- capital using* $\widehat{b}$*.* $S_n^*$ *- capital using* $b^*_{X|Y}$*.*

Physics:

- $X$, $Y$ discrete r.v. Unknown measurement error $P_{Y|X}$.

# Consequence 1 - Universal Engine

Gambling:

- $X$, $Y$ discrete r.v. $P_{X,Y}$ is unknown.
- $\widehat{b}$ - the *universal portfolio* [Cover and Ordentlich 1996].

**Theorem**

$$\lim_{n\to\infty} \frac{1}{n} E[\log \widehat{S}_n - \log S_n^*] = 0.$$

*$\widehat{S}_n$ - capital using $\widehat{b}$. $S_n^*$ - capital using $b^*_{X|Y}$.*

Physics:

- $X$, $Y$ discrete r.v. Unknown measurement error $P_{Y|X}$.
- We developed the analogous *universal control protocol*.

# Consequence 1 - Universal Engine

Gambling:

- $X$, $Y$ discrete r.v. $P_{X,Y}$ is unknown.
- $\widehat{b}$ - the *universal portfolio* [Cover and Ordentlich 1996].

**Theorem**

$$\lim_{n\to\infty} \frac{1}{n} E[\log \widehat{S}_n - \log S_n^*] = 0.$$

$\widehat{S}_n$ *- capital using* $\widehat{b}$*.* $S_n^*$ *- capital using* $b^*_{X|Y}$*.*

Physics:

- $X$, $Y$ discrete r.v. Unknown measurement error $P_{Y|X}$.
- We developed the analogous *universal control protocol*.

**Theorem**

$$\lim_{n\to\infty} \frac{1}{n} E[\widehat{W}_n - W_n^*] = 0.$$

# Consequence 2 - Memory

Gambling:

- Dependent races. At round $i$, $X^{i-1}$, $Y^i$ are known.

# Consequence 2 - Memory

Gambling:

- Dependent races. At round $i$, $X^{i-1}$, $Y^i$ are known.

**Theorem [Permuter, Kim and Weissman 2011]**

$$b^*_{X_i|X^{i-1},Y^i} = P_{X_i|X^{i-1},Y^i}.$$
$$E[\log S_n(X^n||Y^n)] - E[\log S_n(X^n)] = I(Y^n \to X^n).$$

$S_n(X^n||Y^n)$ *- capital with causal side information.*
$S_n(X^n)$ *- capital without side information.*

# Consequence 2 - Memory

Gambling:

- Dependent races. At round $i$, $X^{i-1}$, $Y^i$ are known.

**Theorem [Permuter, Kim and Weissman 2011]**

$$b^*_{X_i|X^{i-1},Y^i} = P_{X_i|X^{i-1},Y^i}.$$
$$E[\log S_n(X^n||Y^n)] - E[\log S_n(X^n)] = I(Y^n \to X^n).$$

$S_n(X^n||Y^n)$ *- capital with causal side information.*
$S_n(X^n)$ *- capital without side information.*

Physics:

- Dependence through fast cycles, hysteresis, etc.

# Consequence 2 - Memory

Gambling:

- Dependent races. At round $i$, $X^{i-1}$, $Y^i$ are known.

**Theorem [Permuter, Kim and Weissman 2011]**

$$b^*_{X_i|X^{i-1},Y^i} = P_{X_i|X^{i-1},Y^i}.$$
$$E[\log S_n(X^n||Y^n)] - E[\log S_n(X^n)] = I(Y^n \to X^n).$$

$S_n(X^n||Y^n)$ - *capital with causal side information.*
$S_n(X^n)$ - *capital without side information.*

Physics:

- Dependence through fast cycles, hysteresis, etc.

**Theorem**

$$Q^*_{X_i|X^{i-1},Y^i} = P_{X_i|X^{i-1},Y^i}.$$
$$E[W_n(X^n||Y^n) - W_n(X^n)] = k_B T I(Y^n \to X^n).$$

# Summary

- One-to-one mapping between gambling and measurement-based work extraction.

# Summary

- One-to-one mapping between gambling and measurement-based work extraction.
- Extension of gambling to continuous random variables.

# Summary

- One-to-one mapping between gambling and measurement-based work extraction.
- Extension of gambling to continuous random variables.
- Universal work extraction.

# Summary

- One-to-one mapping between gambling and measurement-based work extraction.
- Extension of gambling to continuous random variables.
- Universal work extraction.
- Analysis of work extraction with cycle dependence.

# Summary

- One-to-one mapping between gambling and measurement-based work extraction.
- Extension of gambling to continuous random variables.
- Universal work extraction.
- Analysis of work extraction with cycle dependence.
- Full version available on arXiv: `http://arxiv.org/abs/1404.6788`

# Future Work

- Multiple particles.

# Future Work

- Multiple particles.
- Generalize universal portfolios for continuous r.v.

# Future Work

- Multiple particles.
- Generalize universal portfolios for continuous r.v.

Thank you !

# Future Work

- Multiple particles.
- Generalize universal portfolios for continuous r.v.

Thank you !

# Future Work

- Multiple particles.
- Generalize universal portfolios for continuous r.v.

Thank you !

# Future Work

- Multiple particles.
- Generalize universal portfolios for continuous r.v.

Thank you !

# Maximization Over $P_X$

In the Szilard engine:

- Control over $P_X$ through $V_0$.
- Maximal extracted work is

$$\max E[W] = k_B T \max_{P_X} I(X;Y).$$

# Maximization Over $P_X$

In the Szilard engine:

- Control over $P_X$ through $V_0$.
- Maximal extracted work is

$$\max E[W] = k_B T \max_{P_X} I(X;Y).$$

In gambling:

- Control over $P_X$ through choice of race track.
- Maximal capital growth is

$$\max E[\log S_n] = n \max_{P_X \in \mathcal{P}} I(X;Y).$$