DOUBLE COBOUNDARIES FOR COMMUTING CONTRACTIONS

GUY COHEN AND MICHAEL LIN

Dedicated to the memory of Alexander Rubinov, on his 75th birthday

ABSTRACT. Let T and S be commuting contractions on a Banach space X. We consider the problem of when a vector $x \in X$ is a *double coboundary* – i.e. is of the form $x = (I-T)(I-S)y$ for some $y \in X$. We show that when T and S are uniformly ergodic with the same sets of fixed points, x is a double coboundary if and only if

$$\sup_n \|(\sum_{0\le k\le n-1} S^k)(\sum_{0\le k\le n-1} T^k)x\| < \infty,$$

and deduce a characterization of when T and S are both uniformly ergodic, extending a result of Fonf, Lin and Rubinov for a single contraction. When X is a dual space and the contractions S and T are dual operators, x is a double coboundary if and only if

$$\sup_n \|(\sum_{0\le k\le n-1} S^k)(\sum_{0\le k\le n-1} T^k)x\| < \infty,$$

with no additional assumptions.

1. INTRODUCTION

Gottschalk and Hedlund proved in their book [23, p. 135] that if θ is a minimal homeomorphism of a compact Hausdorff space K (i.e. for every $x \in K$ the orbit $\{\theta^k x\}_{k\ge 0}$ is dense in K), then a continuous function f is of the form $f = g - g\circ\theta$ for some continuous g if (and only if) $\sup_n \|\sum_{k=0}^{n-1} f\circ\theta^k\|_{C(K)} < \infty$. Browder [7] proved that if T is a power-bounded operator in a reflexive Banach space X, then

$$(1.1)\qquad x \in (I-T)X \quad \text{if and only if} \quad \sup_n \|\sum_{k=0}^{n-1} T^k x\| < \infty.$$

Browder's result was rediscovered in [9]. Lin [36, Theorem 3.1] (see also Lin and Sine [37]) extended Browder's result to the case that T is a dual operator in a dual Banach space; Browder's result is then a corollary. Additional information is given in [16, Part 2]. It was also proved in [37] that every contraction in L_1 satisfies (1.1). Note that by [21], there is a power-bounded T on L_1 for which (1.1) fails. Wittmann (see [31]) proved that if T is the Markov operator induced on the space of bounded measurable functions by

2010 *Mathematics Subject Classification.* Primary: 47A35; Secondary; 47A50, 28D05, 37A05.

Key words and phrases. linear contractions, coboundaries, cohomology equation, Poisson's equation, Markov operators.

Research of the first author was supported by the Israel Science Foundation (grant No. 1/12).

a transition probablity, then (1.1) holds. The result of [23] was extended to irreducible Markov operators on $C(K)$ of a compact Hausdorff space K by Kornfeld and Lin [31].

Unaware of Browder's result, Robinson [41] proved (1.1) when X is a Hilbert space and T is unitary (using the spectral theorem). Leonov [34] gave a proof of (1.1) for contractions in a Hilbert space (although his statement is for isometries) satisfying a "mixing condition". It was later noted by Aaronson and Weiss [1] that Leonov's proof yields (1.1) when $X = L_p(\mu)$, $1 < p < \infty$, and T is induced by a transformation preserving the probability μ; it is also indicated in [1] how to modify Leonov's proof to obtain the result when $p = 1$. Kozma and Lev [32, Theorem 4.1] noticed that Robinson's proof yields that for T unitary on a Hilbert space H, the condition $\sup_N \frac{1}{N}\sum_{n=1}^{N} \| \sum_{k=0}^{n-1} T^k x\|^2 < \infty$ is sufficient (and obviously necessary) for $x \in (I - T)H$, and gave an even weaker sufficient condition. Conditions for solving the equation $f = h - h \circ \theta$ when θ is a probability preserving transformation and f is a given measurable function were given in [2], [42] and [43]; Anosov [3] proved that if f is integrable, then $\int f = 0$ (even if h is not integrable). As an example, Anosov proved that for θ an irrational rotation of the circle there exists a continuous function $f = h - h \circ \theta$ with h not integrable; see also [29]. On the other hand, it follows from the work of Quas [40, Theorem 1] that if f is continuous and $f = h - h \circ \theta$ with $h \in L_\infty$, then there exists g continuous such that $f = g - g \circ \theta$. For some additional results on coboundaries of rotations see [44],[24],[6].

Elements of the linear manifold $(I - T)X$ are called *coboundaries.* The equation $(I - T)y = x$ with x given is called the *cohomology equation* in ergodic theory, and the (discrete) *Poisson equation* in the theory of discrete time Markov processes.

For a bounded representation $T(s)$ of a semi-group $\mathcal{S}$ by linear operators on a Banach space X, we call a function $x(s)$ from $\mathcal{S}$ to X *a coboundary* if there exists $y \in X$ such that $x(s) = (I - T(s))y$ for every $s \in \mathcal{S}$. If $x(s)$ is a coboundary, then

$$x(s_1 s_2) = (I - T(s_1 s_2))y = y - T(s_1)T(s_2)y =$$

$$y - T(s_1)y + T(s_1)(y - T(s_2)y = x(s_1) + T(s_1)x(s_2).$$

A function $x(s)$ satisfying $x(s_1 s_2) = x(s_1) + T(s_1)x(s_2)$ is called a *cocycle.* The above shows that a coboundary is a bounded cocycle. For the representation $\{T^n\}$ of $\mathbb{N}$ by a power-bounded operator T, $x(n)$ is a cocycle if and only if it is of the form $x(n) = \sum_{k=0}^{n-1} T^k x$; when $x = (I - T)y$, then this cocycle equals $(I - T^n)y$.

Moulin-Ollagnier and Pinchon [38] extended the theorem of Gottschalk and Hedlund to group actions by homeomorphisms of a compact Hausdorff space, proving that a bounded cocycle of a minimal group action is a coboundary. Browder's theorem was extended by Parry and Schmidt [39] to bounded representations of a LCA group G by linear operators in a reflexive Banach space: *a bounded cocycle is a coboundary.* Unaware of [38], Kornfeld and Lin [31] proved the result of [38] for minimal actions of semi-groups, and extended it to irreducible Markov representations in $C(K)$ of certain semi-groups.

In this paper we deal with two commuting contractions T and S on a Banach space X. The partial double sums $\sum_{\ell=0}^{m-1}\sum_{j=0}^{n-1} S^jT^\ell x$ are not a cocycle (of the $\mathbb{N}^2$ representation), but are uniformly bounded if $x = (I-T)(I-S)y$ for some $y \in X$. We call such an x a *double coboundary* (for T and S). Similarly, for d commuting contractions $T_1, \dots, T_d$ we call $x \in X$ a *d-tuple coboundary* if $x = [\Pi_{j=1}^d (I-T_j)]y$ for some $y \in X$. The problem of when x is a d-tuple coboundary of duals in L_p $(1 \le p < \infty)$ of d commuting probability preserving transformations was studied by Bradley [5] and Gordin [22]; El Machkouri and Giraudo [18] gave conditions for representing a function as a sum of a martingale and sums of multiple coboundaries of subsets of $\{T_1, \dots, T_d\}$ (with the goal of obtaining some central limit theorems, by martingale approximation).

In Section 2 we characterize uniform ergodicity of d commuting contractions with equal sets of fixed points (e.g. commuting ergodic Markov operators) by the set of d-tuple coboundaries being closed. Fonf, Lin and Rubinov [20, Theorem 1.1] observed that a contraction T is uniformly ergodic if and only if the set of vectors x with bounded partial sums (i.e. $\sup_n \|sum_{k=0}^n T^k x\| < \infty$) is closed. For d commuting contractions with equal sets of fixed points we obtain a d-dimensional extension of the above result of [20] when X is reflexive, and in the general case we prove it under an additional mild "ergodicity" assumption (which is automatically satisfied by a single contraction).

In Section 3 we consider the problem of when boundedness of the partial double sums (with respect to two commuting contractions S and T) of a vector x implies that x is a double coboundary. This is an extension in a different direction of the results of Gottschalk-Hedlund and Browder. Bradley [5, Corollary 2.2] proved (using a different terminology) that for d commuting probability preserving invertible transformations, boundedness in L_p $(1 \le p \le \infty)$ of the partial d-multiple sums of $f \in L_p$ is equivalent to f being a d-tuple coboundary. His proof cannot be used in the general context of power-bounded operators on reflexive Banach spaces.

2. Multiple coboundaries and uniform ergodicity

It is well-known (e.g. [33, p. 73]) that if T is a power-bounded operator on a Banach space X (not necessarily reflexive), then $\|\frac{1}{n}\sum_{k=0}^n T^k x\| \to 0$ if and only if $x \in \overline{(I-T)X}$. The set of $x \in X$ such that $\frac{1}{n}\sum_{k=0}^n T^k x$ converges is a closed subspace which equals $F(T) \oplus \overline{(I-T)X}$, where $F(T)$ is the set of fixed points of T; when X is reflexive we have the *ergodic decomposition* $X = F(T) \oplus \overline{(I-T)X}$. When the ergodic decomposition holds for T power-bounded (in a not necessarily reflexive space), T is called *mean ergodic.* If the convergence of $\frac{1}{n}\sum_{k=0}^n T^k$ is uniform on the unit ball of X, we call T *uniformly ergodic.*

In this section we obtain an ergodic decomposition for d commuting mean ergodic power-bounded operators, using d-tuple coboundaries. The decomposition is used to characterize the d-tuple coboundaries of commuting uniformly ergodic power-bounded operators with common fixed points.

Proposition 2.1. *Let $T_1, \dots, T_d$ be commuting mean ergodic power-bounded operators on a Banach space X. Then the linear manifold*

$$Y := \{x \in X : x = \sum_{j=1}^{d} z_j + [\Pi_{j=1}^{d}(I - T_j)]y \quad y \in X,\ T_j z_j = z_j\}$$

is dense in X.

Proof. For $d = 1$ this follows from the decomposition $X = F(T) \oplus \overline{(I - T)X}$.

We now prove for $d = 2$. By commutativity, $T_1 z_1 = z_1$ implies that $T_1(T_2 z_1) = T_2 z_1$. The ergodic decomposition yields that $F(T_2) + (I - T_2)X$ is dense, and for $z_2 + (I - T_2)y_2$ we approximate y_2 by $z + (I - T_1)y$ with $z \in F(T_1)$. Then $z_2 + (I - T_2)(z + (I - T_1)y) = z_2 + (z - T_2 z) + (I - T_2)(I - T_1)y$ yields the result with $z_1 = z - T_2 z$.

For $d > 2$ we proceed by induction, using similarly the ergodic decomposition of T_d. □

Lemma 2.2. *Let $T_1, \dots, T_d$ be commuting mean ergodic power-bounded operators on a Banach space X. Then*

$$X = \Big[\bigcap_{j=1}^{d} F(T_j)\Big] \oplus \overline{\sum_{j=1}^{d}(I - T_j)X}. \tag{2.1}$$

Proof. Denote $A_n(T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k$ and $M_n := \Pi_{j=1}^{d} A_n(T_j)$. Since $A_n(T_j)$ converges strongly as $n \to \infty$ to a projection P_j on $F(T_j)$ which annihilates $\overline{(I - T_j)X}$, by commutativity $M_n x$ converges strongly to $\Pi_{j=1}^{d} P_j$, which is a projection on $\bigcap_{j=1}^{d} F(T_j)$ which annihilates all the images $(I - T_j)X$, and we get the decomposition as in the case of a single operator. □

Remark. The lemma follows also from the general Koliha-Nagel-Sato decomposition (see [33, p. 79]), but for that we first show that the sum of the ranges of the $(I - T_j)$ contains all the ranges $(I - \Pi_{j=1}^{d} T_j^{k_j})X$. This proof is not shorter.

Theorem 2.3. *Let $T_1, \dots, T_d$ be commuting mean ergodic power-bounded operators on a Banach space X. Then*

$$X = \overline{\sum_{j=1}^{d} F(T_j)} \oplus \overline{[\prod_{j=1}^{d}(I - T_j)]X}. \tag{2.2}$$

Proof. We use the notations of the previous lemma, $E_j := I - \lim_n A_n(T_j) = I - P_j$ is well-defined in the strong operator topology, and is a projection on $\overline{(I - T_j)X}$ with null space $F(T_j)$. Then $F(E_j) = \overline{(I - T_j)X}$ and $(I - E_j)X = P_j X = F(T_j)$. Applying the previous lemma to $E_1, \dots, E_d$ we obtain

$$X = \Big[\bigcap_{j=1}^{d} \overline{(I - T_j)X}\ \Big] \oplus \overline{\sum_{j=1}^{d} F(T_j)}.$$

Now the set of d-tuple coboundaries $\big[\Pi_{j=1}^{d}(I - T_j)\big]X$ is a subset of the left summand above, and by the density given in Proposition 2.1 we get the assertion. □

Remark. It is clear that $\overline{[\prod_{j=1}^d (I-T_j)]X} \subset \overline{\sum_{j=1}^d (I-T_j)X}$. When the commuting mean ergodic operators have the same set of fixed points (e.g. induced by commuting ergodic measure preseving transformations on a probability space $(\mathbb{S},\Sigma,\mu)$), the decompositions (2.1) and (2.2) yield that the two closed subspaces above are equal. Moreover, for each j the ergodic decomposition for T_j then shows that these subspaces equal $\overline{(I-T_j)X}$.

Definition. A vector $x \in X$ is called a *mixed coboundary* for the (commuting) transformations $T_1,\dots,T_d$ if $x \in \sum_{j=1}^d (I-T_j)X$.

Mixed coboundaries were used in [11]; a characterization of mixed coboundaries of commuting unitary operators with countable Lebesgue spectrum was given (for $d=2$) in [12, p. 13].

Theorem 2.4. *Let $T_1,\dots,T_d$ be commuting mean ergodic power-bounded operators on a Banach space X with $F(T_j)=F(T_1)$ for $1\le j\le d$. Then the following are equivalent:*

(i) All the T_j are uniformly ergodic.

(ii) Every mixed coboundary is a d-tuple coboundary.

(iii) $[\prod_{j=1}^d (I-T_j)]X = (I-T_k)X$ for every $1\le k\le d$.

Proof. (i) $\Longrightarrow$ (ii): By the previous remark, for every k we have

$$\overline{[\prod_{j=1}^d (I-T_j)]X} = \overline{\sum_{j=1}^d (I-T_j)X} = \overline{(I-T_k)X}.$$

By uniform ergodicity, $(I-T_k)X$ is closed, and $(I-T_k)$ is invertible on it. Since these subspaces are the same, denoted by X_0, all the $I-T_k$ are invertible on X_0, so $\prod_{j=1}^d (I-T_j)$ is invertible on X_0. Thus, if $x = \sum_{k=1}^d (I-T_k)y_k$, then $(I-T_k)y_k = [\prod_{j=1}^d (I-T_j)]z_k$, so x is a d-tuple coboundary.

(ii) $\Longrightarrow$ (iii): Fix k and let $x=(I-T_k)y$. Then x is a mixed coboundary, and by (ii) $x=[\prod_{j=1}^d (I-T_j)]z$, so

$$(I-T_k)X \subset [\prod_{j=1}^d (I-T_j)]X \subset (I-T_k)X,$$

so equality holds and (iii) follows.

(iii) $\Longrightarrow$ (i): Fix k and assume that T_k is not uniformly ergodic. Then $(I-T_k)X$ is not closed, and we take $y\in\overline{(I-T_k)X}$ which is not in $(I-T_k)X$. Put $x=(I-T_k)y$. By (iii) there is a z such that $x=[\prod_{j=1}^d (I-T_j)]z$, so

$$(I-T_k)\big(y - [\prod_{j\ne k}(I-T_j)]z\big) = 0. \tag{2.3}$$

By (iii) $(I-T_j)X=(I-T_k)X$ for $j\ne k$, hence $y\in\overline{(I-T_j)X}$, so $y-[\prod_{\ell\ne k}(I-T_\ell)]z$ is in $\overline{(I-T_j)X}=\overline{(I-T_k)X}$; since it is in $F(T_k)$ by (2.3), it is zero. Hence $y\in(I-T_j)X$ for $j\ne k$, so by (iii) $y\in(I-T_k)X$ – a contradiction. Hence $(I-T_k)X$ is closed, so T_k is uniformly ergodic [35]. □

Remarks. 1. Without the assumption that $F(T_j) = F(T_k)$ for $j \neq k$, (i) need not imply (ii). Let $X \neq \{0\}$ be a finite dimensional Hilbert space, let $0 \neq T_1 \neq I$ be an orthogonal projection, and define $T_2 = I - T_1$. Then the mixed coboundaries are all of X, but 0 is the only double coboundary.

2. Under the assumptions on $T_1, \ldots, T_d$, the equality $(I - T_j)X = (I - T_1)X$ for every j is not sufficient for uniform ergodicity – take T_1 not uniformly ergodic and $T_j = T_1$ for $j > 1$.

3. A non-trivial example can be obtained using Proposition 2.2 of [14]: Let T be the unitary operator induced on L_2 by an invertible ergodic measure preserving transformation on a non-atomic probability space, and define $T_j = \sum_{n=-\infty}^{\infty} p_n^{(j)} T^n$, where

$$p_n^{(j)} > 0, \quad \sum_{n=-\infty}^{\infty} p_n^{(j)} = 1, \quad \sum_{n=-\infty}^{\infty} |n| p_n^{(j)} < \infty, \quad \sum_{n=-\infty}^{\infty} n p_n^{(j)} \neq 0.$$

Then $F(T_j) = F(T)$ by uniform convexity of L_2, and by [14] $(I - T_j)L_2 = (I - T)L_2$ for every j. Since the spectrum of T is the whole unit circle, 1 is not isolated in the spectrum of T_j, so T_j is not uniformly ergodic.

4. On the other hand, if T and S are induced on L_2 by commuting ergodic invertible probability preserving transformations as above and $(I - T)L_2 = (I - S)L_2$, then by the result of Kornfeld [30], $T = S^{\pm 1}$.

Lemma 2.5. *Let $T_1, \ldots, T_d$ be commuting power-bounded operators on a Banach space X with $F(T_j) = F(T_1)$ for $1 \leq j \leq d$. Then*

$$\{x \in X : [\prod_{j=1}^{d} (I - T_j)]x = 0\} = F(T_1) \tag{2.4}$$

Proof. The inclusion of the right-hand side of (2.4) in its left-hand side is trivial. The proof of equality is by induction on the number of operators. For $d = 1$ this is the definition of $F(T_1)$. Assume the assertion is true for $d - 1$ operators, $d > 1$.

Let $[\prod_{j=1}^{d}(I - T_j)]x = 0$. Then $[\prod_{j=1}^{d-1}(I - T_j)]x \in F(T_d) = F(T_1)$; but it is obviously also in $\overline{(I - T_1)X}$, so $[\prod_{j=1}^{d-1}(I - T_j)]x = 0$, and by the induction hypothesis $x \in F(T_1)$. □

Theorem 2.6. *Let $T_1, \ldots, T_d$ be commuting power-bounded operators on a Banach space X with $F(T_j) = F(T_1)$ for $1 \leq j \leq d$. Then the set of d-tuple coboundaries $[\prod_{j=1}^{d}(I - T_j)]X$ is closed if and only if all the T_j are uniformly ergodic.*

Proof. We denote $Y := [\prod_{j=1}^{d}(I - T_j)]X$. The case $d = 1$ is proved in [35], so we assume $d > 1$.

Assume that Y is closed. We use the ideas of [35] to prove uniform ergodicity of all T_j. The operator $M = \prod_{j=1}^{d}(I - T_j)$ maps X onto the closed subspace Y, so by the open mapping theorem (e.g. [17, p. 487]) there is $K > 0$ such that for $y \in Y$ there exists $x \in X$ with $\|x\| \leq K\|y\|$ and $Mx = y$. Denote by $\hat{T}_j$ the restriction of T_j to the invariant subspace

Y. Then for $y \in Y$ we have

$$\|A_n(T_1)y\| \le \frac{\|I - T_1^n\|}{n} \Big\| \prod_{j=2}^d (I - T_j) \Big\| \|x\| \le \frac{\|I - T_1^n\|}{n} \Big\| \prod_{j=2}^d (I - T_j) \Big\| K \|y\|.$$

Hence $\big\|A_n(\hat{T}_1)\big\| \to 0$ (on Y). Since $I - A_n(\hat{T}_1) = \frac{I - \hat{T}_1}{n} \sum_{k=1}^{n-1} \sum_{\ell=0}^{k-1} \hat{T}_1^\ell$, when $\|A_n(\hat{T}_1)\| < 1$ we have that $I - \hat{T}_1$ is invertible on Y. Similarly, all $I - \hat{T}_j$ are invertible on Y, so $Y = [\prod_{j=1}^d (I - \hat{T}_j)]Y$. Thus for $x \in X$ there is a $z \in Y$ with $Mx = Mz$, so $M(x - z) = 0$, and by the previous lemma $x - z \in F(T_1)$. We therefore obtain the ergodic decomposition $X = F(T_1) \oplus Y \subset F(T_1) \oplus \overline{(I - T_1)X}$ (note that the decompositions (2.1) and (2.2) require mean ergodicity, which was *not assumed*). Hence

$$(I - T_1)X \subset \overline{(I - T_1)X} = Y \subset (I - T_1)X,$$

which shows that $(I - T_1)X = Y$ is closed, so T_1 is uniformly ergodic by [35]. Similarly all T_j are uniformly ergodic.

Assume now that each T_j is uniformly ergodic. As remarked above, the mean ergodicity and the fact that all the operators have the same fixed points imply that $\overline{Y} = \overline{(I - T_j)X}$ for every j. By uniform ergodicity $(I - T_j)X$ is closed and $(I - T_j)$ is invertible on $(I - T_j)X = \overline{Y}$ [35]. This yields that $\prod_{j=1}^d (I - T_j)$ is invertible on $\overline{Y}$, so $\overline{Y} = [\prod_{j=1}^d (I - T_j)]\overline{Y} \subset Y$, which shows that Y is closed. □

Example. *Commuting uniformly ergodic contractions with common fixed points.*

Let μ be a probability measure on the Borel sets of the unit circle $\mathbb{T}$. Fix $1 \le p < \infty$, and on $L_p(\mathbb{T}, \lambda)$ (where λ is the normalized Haar measure) define $Tf = \mu * f$. Then T is a Markov operator with invariant probability λ, and $\|Tf - \int f\, d\lambda\|_p \le \|\mu - \lambda\| \cdot \|f\|_p$ (where the norm of a measure is its total variation). Let $\mu_1, \dots, \mu_d$ be probabilities on $\mathbb{T}$ with corresponding operators $T_1, \dots, T_d$. Then $T_j T_k$ corresponds to convolution by $\mu_j * \mu_k$, which commute since the unit circle is an Abelian group. If the μ_j are all absolutely continuous, then $F(T_j)$ consists precisely of the constant functions, and by Theorem 3 of Bhattacharya [4] we have $\|\mu_j^{*n} - \lambda\| \to 0$, which yields that $\|T_j^n - E\|_p \to 0$ (where $Ef = \int f\, d\lambda$).

Note that $Tf = \mu * f$ in $L_2(\mathbb{T}, \lambda)$ is uniformly ergodic if and only if the Fourier-Stieltjes coefficients of μ satisfy $\inf_{n \ne 0} |1 - \hat{\mu}(n)| > 0$ (e.g. [15]).

Theorem 2.7. *Let $T_1, \dots, T_d$ be commuting uniformly ergodic power-bounded operators on a Banach space X with $F(T_j) = F(T_1)$ for $1 \le j \le d$. Then the following are equivalent for $x \in X$:*

(i) $x \in [\prod_{j=1}^d (I - T_j)]X$.

(ii) $\sup_{n_1, n_2, \dots, n_d > 0} \big\| [\prod_{j=1}^d (\sum_{k=0}^{n_j - 1} T_j^k)] x \big\| < \infty$.

(iii) $\sup_{n > 0} \big\| [\prod_{j=1}^d (\sum_{k=0}^{n-1} T_j^k)] x \big\| < \infty$.

Proof. Clearly (i) implies (ii) and (ii) implies (iii).

Assume (iii), and let Y be as in the previous proof. By Theorem 2.6 Y is closed. Since the operators are uniformly ergodic with the same sets of fixed points, they have the same ergodic decomposition with $(I - T_j)X = Y$.

Hence $\lim_{n\to\infty} A_n(T_j)x$ is independent of j, and we denote it by y, which is T_j-invariant. Then

$$\big\|[\prod_{j=1}^d A_n(T_j)]x - y\big\| = \big\|[\prod_{j=2}^d A_n(T_j)](A_n(T_1)x - y)\big\| \to 0.$$

But by (iii) $\big\|[\prod_{j=1}^d A_n(T_j)]x\big\| \to 0$, so $y = 0$. Thus $A_n(T_1)x \to 0$, and the ergodic decomposition $X = F(T_1) \oplus Y$ yields that $x \in Y$. □

Remark. For any $T_1, \dots, T_d$ commuting power-bounded operators on X,

$$[\prod_{j=1}^d (I - T_j)]X \subset \{x : \sup_{n>0} \big\|[\prod_{j=1}^d (\sum_{k=0}^{n-1} T_j^k)]x\big\| < \infty\}. \tag{2.5}$$

When all the T_j are uniformly ergodic with $F(T_j) = F(T_1)$ for $1 \le j \le d$, Theorems 2.7 and 2.6 yield equality in (2.5), and this linear manifold is closed. Conversely, when $F(T_j) = F(T_1)$ for $1 \le j \le d$ and the left hand side of (2.5) is closed, Theorem 2.6 yields that all the T_j are uniformly ergodic, so we have equality in (2.5) by Theorem 2.7.

A natural question is whether commuting $T_1, \dots, T_d$ with $F(T_j) = F(T_1)$ must be uniformly ergodic when $\{x : \sup_{n>0} \big\|[\prod_{j=1}^d (\sum_{k=0}^{n-1} T_j^k)]x\big\| < \infty\}$ is closed. The answer is positive when X is reflexive, since by Theorem 3.1 in the next section we have equality in (2.5), and then Theorem 2.6 applies. A partial (positive) answer is given in the next theorem.

Theorem 2.8. *Let $T_1, \dots, T_d$ be commuting power-bounded operators on a Banach space X with $F(T_j) = F(T_1)$ for $1 \le j \le d$, and assume that for $x \in \overline{(I - T_1)X}$ we have*

(*) $\frac{1}{n} \sum_{k=0}^n R^k x \to 0$ for every $R \ne I$ of the form $R = \prod_{j=1}^d T_j^{\epsilon_j}$ with ϵ_j zero or one.

If $\{x : \sup_{n>0} \big\|[\prod_{j=1}^d (\sum_{k=0}^{n-1} T_j^k)]x\big\| < \infty\}$ is closed, then all the T_j are uniformly ergodic.

Proof. Denote $Y := \overline{[\prod_{j=1}^d (I - T_j)]X}$. Then Y is invariant for all the T_j, and we denote by $\hat{T}_j$ the restriction of T_j to Y. Put $S_n := \prod_{j=1}^d \big(\sum_{k=0}^{n-1} T_j^k\big)$ and similarly define $\hat{S}_n$.

If $y = \prod_j (I - T_j)x$, then for each k we have $\|A_n(T_k)y\| \to 0$ as $n \to \infty$, which shows that $\hat{T}_k$ is mean ergodic (on Y). Hence $\overline{(I - \hat{T}_k)Y} = Y$, and by the remark following Theorem 2.3 applied to Y, also $Y = \overline{\big[\prod_{j=1}^d (I - \hat{T}_j)\big]Y}$.

By (2.5) and the assumption, $Y \subset \{x \in X : \sup_n \|S_n x\| < \infty\}$, so $\sup_n \|\hat{S}_n\| < \infty$ by the Banach-Steinhaus theorem.

If $y = \prod_j (I - T_j)x$ with $x \in Y$, then $\hat{S}_i y = S_i y = \prod_{j=1}^d (I - T_j^i)x$. Then

$$\frac{1}{n} \sum_{i=1}^n \hat{S}_i y = x + \sum_\ell \pm \frac{1}{n} \sum_{i=1}^n R_\ell^i x \to_{n\to\infty} x \tag{2.6}$$

by assumption (*), where ℓ goes over all non-zero $(\epsilon_1, \dots, \epsilon_d) \in \{0,1\}^d$. Since $\sup_n \|\hat{S}_n\| < \infty$, we obtain that $\tilde{S}_n y := \frac{1}{n} \sum_{i=1}^n \hat{S}_i y$ converges strongly on Y.

Now let $y \in Y$ and let $x_k \in Y$ with $\lim_k [\prod_{j=1}^d (I-\hat{T}_j)]x_k = y$. Put $z = \lim_n \tilde{S}_n y$. Since $\tilde{S}_n x_k \to x_k$ as $n \to \infty$, we have

$$\|z - x_k\| = \lim_n \|\tilde{S}_n y - x_k\| \le$$

$$\limsup_n \|\tilde{S}_n y - \tilde{S}_n \prod_j (I-\hat{T}_j)x_k\| + \lim_n \|x_k - \tilde{S}_n \prod_j (I-\hat{T}_j)x_k\| \le$$

$$K\|y - \prod_j (I-\hat{T}_j)x_k\|,$$

which shows that $x_k \to z$, hence $y = [\prod_{j=1}^d (I-\hat{T}_j)]z$. We have obtained that

$$Y = [\prod_{j=1}^d (I-\hat{T}_j)]Y \subset [\prod_{j=1}^d (I-T_j)]X \subset Y$$

so $[\prod_{j=1}^d (I-T_j)]X$ is closed, and Theorem 2.6 yields that all the T_j are uniformly ergodic. □

Remarks. 1. When $d=1$, condition (*) is automatically satisfied, and we obtain that if T is power-bounded with $\{x \in X : \sup_n \|\sum_{k=0}^n T^k x\| < \infty\}$ closed, then T is uniformly ergodic. This was observed by Fonf, Lin and Rubinov in [20, Theorem 1.1].

2. For commuting T_j as in the theorem, condition (*) is satisfied when $T_j^n(I-T_j)$ converges strongly to zero for each j.

3. Double coboundaries of dual power-bounded operators in dual spaces

Let T and S be commuting power-bounded operators on a Banach space X. Then $|||x||| := \sup_{j,k\ge 0} \|S^j T^k x\|$ is an equivalent norm on X for which T and S are contractions. For brevity, we therefore state our results below for contractions, but they apply to power-bounded operators as well.

Theorem 3.1. *Let T and S be commuting contractions on a reflexive Banach space X. Then the following are equivalent for $x \in X$:*

(i) $\sup_{n,m\ge 1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} S^j T^\ell x\| < \infty$.

(ii) There exists $z \in X$ such that $x = (I-S)(I-T)z$.

(iii) $\sup_{n\ge 1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^j T^\ell x\| < \infty$.

Proof. Clearly (ii) implies (i). Obviously (i) implies (iii).

Assume (iii). Define $R_n = \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^j T^\ell = (\sum_{j=0}^{n-1} S^j)(\sum_{\ell=0}^{n-1} T^\ell)$. Then

$$R_{n+1} - TSR_n = (I + \sum_{j=1}^n S^j)(I + \sum_{\ell=1}^n T^\ell) - (\sum_{j=1}^n S^j)(\sum_{\ell=1}^n T^\ell) =$$

$$I + \sum_{j=1}^n S^j + \sum_{\ell=1}^n T^\ell.$$

Hence (iii) yields $\sup_n \|\sum_{j=1}^n S^j x + \sum_{\ell=1}^n T^\ell x\| \le \|x\| + 2\sup_n \|R_n x\| < \infty$, which yields

$$\|\frac{1}{n}\sum_{j=1}^n S^j x + \frac{1}{n}\sum_{\ell=1}^n T^\ell x\| \to 0. \tag{3.1}$$

Put $y_n = \frac{1}{n}\sum_{k=1}^n \sum_{0\le j,\ell\le k-1} S^j T^\ell x$ and

$$x_n = (I-S)(I-T)y_n = (I-S)(I-T)\big[\frac{1}{n}\sum_{k=1}^n \sum_{0\le j,\ell\le k-1} S^j T^\ell x\big]. \tag{3.2}$$

Since (y_n) is assumed bounded, by weak sequential compactness there is a subsequence (y_{n_j}) which converges weakly, say to y, and then $x_{n_j} \to (I-S)(I-T)y$ weakly. But

$$x_n = \frac{1}{n}\sum_{k=1}^n (I-S^k)(I-T^k)x = x - \frac{1}{n}\sum_{k=1}^n S^k x - \frac{1}{n}\sum_{k=1}^n T^k x + \frac{1}{n}\sum_{k=1}^n (ST)^k x.$$

Put $v = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n (TS)^k x$. Using (3.1) we obtain $x_n \to x + v$, so $(I-S)(I-T)y = x + v$. Put $u = \lim_n \frac{1}{n}\sum_{k=1}^n (TS)^k y$. Since $(TS)v = v$, we obtain

$$2v = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n (TS)^k (x+v) = \lim_{n\to\infty} (I-S)(I-T)\frac{1}{n}\sum_{k=1}^n (TS)^k y =$$

$$(I-S)(I-T)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n (TS)^k y = (I-S)(I-T)u.$$

Hence $x = (I-S)(I-T)y - v = (I-S)(I-T)(y - \frac{1}{2}u)$, so (ii) holds with $z = y - \frac{1}{2}u$. □

Remarks. 1. Given a power-bounded operator T on X reflexive, if $x \in X$ satisfies $\sup_{n>0} \|\sum_{k=1}^n T^k x\| < \infty$, then x is a coboundary: by Browder's theorem $Tx = (I-T)z$ for some z, and then $x = (I-T)(x+z)$. For two operators, if in (i) of the theorem we start the summation from 1 instead of 0, the condition need not imply that x is a double coboundary. For example, let $X = \mathbb{R}^3$, $T(a,b,c) = (a,0,0)$, and $S(a,b,c) = (0,b,0)$. It is easily checked that $(I-T)(I-S)(a,b,c) = (0,0,c)$, so the double cobundaries are the set $\{(0,0,c) : c \in \mathbb{R}\}$. However, $\sum_{j=1}^m \sum_{\ell=1}^n S^j T^\ell (a,b,c) = (0,0,0)$ for every $(a,b,c) \in X$, since $TS = 0$.

2. Note that once (ii) is proved, we see that the whole sequence $\{y_n\}$ defined in the above proof actually converges in norm.

Example. *Fourier characterization of double coboundaries for the "shifts" on $L_2(\mathbb{T}^2)$.*

Let $X = L_2(\mathbb{T}^2)$ with the normalized Lebesgue measure. We identify the unit circle $\mathbb{T}$ with the interval $[0,2\pi)$, and define on X the operators $Tf(s,t) = \mathrm{e}^{it} f(s,t)$ and $Sf(s,t) = \mathrm{e}^{is} f(s,t)$. Let

$$\widehat{f}(k,j) = \int\int \mathrm{e}^{-ikt}\mathrm{e}^{-ijs} f(s,t)\,dt\,ds$$

be the two-dimensional Fourier coefficients of $f(s,t)$. Then

$$\widehat{Tf}(k,j) = \widehat{f}(k-1,j) \quad \text{and} \quad \widehat{Sf}(k,j) = \widehat{f}(k,j-1),$$

so T and S shift the Fourier coefficients. Computing the double sums we obtain

$$\Big|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^j T^\ell f\Big|^2 = \sum_{0\le j,j',\ell,\ell'<n} \mathrm{e}^{i(j-j')s}\mathrm{e}^{i(\ell-\ell')t}|f(t,s)|^2.$$

Put $g = |f|^2$, and let $\widehat{g}(k,j)$ be its two-dimensional Fourier coefficients. Then

$$\Big\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^j T^\ell f\Big\|^2 = \sum_{0\le j,j',\ell,\ell'<n} \widehat{g}(\ell'-\ell, j'-j).$$

Thus, by Theorem 3.1, $f \in X$ is a double coboundary if and only if

$$\sup_n \sum_{0\le j,j',\ell,\ell'<n} \widehat{|f|^2}(\ell-\ell', j-j') < \infty. \tag{3.3}$$

The above example yields the following application to two-dimensional Fejér means. Let $0 \le g \in L_1(\mathbb{T}^2)$ and denote by $\sigma_{n,m}(g)$ and $\sigma_n(g) = \sigma_{n,n}(g)$ the Fejér means of g along rectangles and squares, respectively (see Zygmund [46, Ch. XVII]).

Corollary 3.2. *Let $0 \le g \in L_1(\mathbb{T}^2)$. Then for every $(s,t) \in \mathbb{T}^2$,*

$$\sup_n n^2\sigma_n(g)(s,t) < \infty \quad \text{if and only if} \quad \sup_{n,m} nm\,\sigma_{n,m}(g)(s,t) < \infty.$$

Proof. First we prove the case $s = t = 0$. Put $f = \sqrt{g}$. For the operators S and T in the example, Theorem 3.1 yields that condition (3.3) is equivalent to

$$\sup_{n,m} \sum_{0\le j,j'<n}\sum_{0\le \ell,\ell'<m} \widehat{|f|^2}(\ell-\ell', j-j') < \infty. \tag{3.4}$$

Putting $S_n(g)(s,t) = \sum_{0\le|k|,|j|<n} \widehat{g}(k,j)\mathrm{e}^{i(jt+ks)}$, we see that (3.3) is equivalent to

$$\sup_n \Big|\sum_{\ell=0}^{n} S_\ell(g)(0,0)\Big| < \infty.$$

By the definition of Fejér means, (3.3) and (3.4) are equivalent to $\sup_n n^2\sigma_n(g)(0,0) < \infty$ and $\sup_{n,m} nm\,\sigma_{n,m}(g)(0,0) < \infty$, respectively. The equivalence (3.3)$\Leftrightarrow$(3.4) yields the case $(0,0)$.

The general case then follows by a suitable translations of the arguments of g. □

Remarks. 1. It is known that for a bounded and continuous function g with $g(0,0) = 0$ we have $\sigma_{n,m}(g)(0,0) \to_{n,m} 0$, [46, Ch. XVII, Th. 1.20]. Without any additional conditions, this convergence does not hold with a rate. However, for a coboundary we obtain a rate.

2. For an L_1 function g, the means along squares $\sigma_n(g)$ converge a.e. [46, Ch. XVII, Th. 3.1] while the means along unrestricted rectangles $\sigma_{n,m}(g)$ may diverge (e.g., see [27]). The above corollary yields that we have a

convergence to zero with a rate $1/n^2$ along squares if and only if we have a convergence with rate $1/nm$ along rectangles.

Lemma 3.3. *Let T and S be commuting contractions on a Banach space X with T^n and S^m converging in the weak operator topology. If there exist infinite increasing sequences $\{m_i\}$ and $\{n_i\}$ such that*

$$\sup_i \| \sum_{j=0}^{n_i-1} \sum_{\ell=0}^{m_i-1} S^j T^\ell x\| = K < \infty, \tag{3.5}$$

then $x \in \overline{(I-T)(I-S)X}$.

Proof. By the assumption T and S are mean ergodic, with weak-lim $S^m = P_1 := \lim A_m(S)$ and weak-lim $T^n = P_2 := \lim A_n(T)$. Since $TS = ST$, also P_1 and P_2 commute, and they also commute with T and S. By Theorem 2.3 $X = \overline{F(S) + F(T)} \oplus \overline{(I-S)(I-T)X}$. Then $P := P_1 + P_2 - P_1P_2$ is the projection on $\overline{F(S) + F(T)}$ corresponding to the above decomposition of X, and P commutes with T and S. Let x satisfy (3.5). Then $P_1P_2x = 0$. Since $P_1S^j = P_1$, application of P_1 to (3.5) yields $\|n_i \sum_{k=0}^{m_i-1} T^k P_1 x\| \le K$. Applying $I-T$ we obtain $\|(I-T^{m_i})P_1x\| \le K\|I-T\|/n_i$. But T^{m_i} converges weakly to P_2, so letting $i \to \infty$ we obtain $P_1x = P_1x - P_2P_1x = 0$. Similarly also $P_2x = 0$, and thus $Px = 0$, which proves the assertion. □

Theorem 3.4. *Let T and S be commuting contractions on a reflexive Banach space, with T^n converging weakly and S^m converging strongly. If there exist infinite increasing sequences $\{m_i\}$ and $\{n_i\}$ such that*

$$\sup_i \| \sum_{j=0}^{n_i-1} \sum_{\ell=0}^{m_i-1} S^j T^\ell x\| < \infty,$$

then $x \in (I-T)(I-S)X$.

Proof. By Lemma 3.3, $x \in \overline{(I-T)(I-S)X}$. By reflexivity, there is a subsequence $\{i_r\}$ such that $\{\sum_{j=0}^{n_{i_r}-1} \sum_{\ell=0}^{m_{i_r}-1} S^j T^\ell x\}$ converges weakly, and we replace $\{n_i\}$ and $\{m_i\}$ by the corresponding subsequences, so we may assume now that $\{\sum_{j=0}^{n_i-1} \sum_{\ell=0}^{m_i-1} S^j T^\ell x\}$ converges weakly, say to y. Since $x \in \overline{(I-T)(I-S)X}$, also y is in that subspace. Let P_1 and P_2 be the ergodic projections corresponding to S and T, as defined in Lemma 3.3. Then

$$(I-T)(I-S)y = \text{weak-}\lim_i (I-T^{n_i})(I-S^{m_i})x =$$

$$x - \text{weak-}\lim_i (T^{n_i}x + S^{m_i}x) + \text{weak-}\lim_i T^{n_i}S^{m_i}x = x - P_2x - P_1x + P_2P_1x.$$

Since $x \in \overline{(I-T)(I-S)X}$, we have $P_1x = P_2x = 0$, so $(I-T)(I-S)y = x$. □

Remark. Putting $S = 0$ we get an improvement of part (c) of the result of Browder and Petryshyn [8] (which holds also for weak convergence).

It is well-known (e.g. [33, p. 65]) that if P is a Markov operator with invariant measure m, then P induces a contraction on each $L_p(m)$, $1 \le p \le \infty$.

Proposition 3.5. *Let P and Q be commuting ergodic Markov operators on a probability space $(\mathbb{S},\Sigma,\mu)$ with μ invariant for both. For $1<p\le\infty$, the following are equivalent for $f\in L_p(\mathbb{S},\Sigma,\mu)$:*

(i) $\sup_{n,m\ge 1}\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1}P^jQ^\ell f\|_p<\infty$.

(ii) There exists $g\in L_p$ with $\int g\,dm=0$ satisfying $f=(I-P)(I-Q)g$.

(iii) $\sup_{n\ge 1}\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1}P^jQ^\ell f\|_p<\infty$.

Proof. Since $\int P^jf\,d\mu=\int Q^\ell f\,d\mu=\int P^jQ^\ell f\,d\mu=\int f\,d\mu$ by invariance of μ, each of the three conditions implies that $\int f\,d\mu=0$. For $1<p<\infty$, take in the previous corollary $X=L_p^0:=\{f\in L_p:\int f\,dm=0\}$.

For $p=\infty$ we have to prove only that (iii) implies (ii). For simplicity we shall assume that $L_1(\mu)$ is separable, so the unit ball of $L_\infty(\mu)$ with the weak-* topology is compact metrizable [17, Theorem V.5.1]. As in the proof of Theorem 3.1, we put $g_n:=\frac{1}{n}\sum_{k=1}^n\sum_{0\le j,\ell\le k-1}S^jT^\ell f$ and $f_n:=(I-P)(I-Q)g_n$. The assumption (iii) yields that (g_n) is bounded in L_∞, so there exists a subsequence (g_{n_j}) which converges weak-* to some $g\in L_\infty(\mu)$. Since μ is a probability, we obtain that $g_{n_j}\to g$ weakly in $L_2(\mu)$, and similarly $L_2-\lim_n\frac{1}{n}\sum_{k=1}^n(PQ)^kf$ is a bounded function. Now the proof of Theorem 3.1 for L_2 yields that $f=(I-P)(I-Q)h$ for some $h\in L_\infty$. □

Remark. Proposition 3.5 applies when P and Q are induced by commuting probability preserving ergodic transformations θ and τ on $(\mathbb{S},\Sigma,\mu)$. For example, let θ and τ be irrational rotations on the unit circle. Another example is obtained when $\theta x=rx \mod 1$ and $\tau x=sx \mod 1$ on $([0,1),\mathcal{B},dx)$ for any pair of positive integers r and s (since $rx \mod 1$ is isomorphic, via expansion by basis r, to the one-sided Bernoulli shift of i.i.d. random variables with equi-probable r outcomes, it is ergodic).

Theorem 3.6. *Let $X=Y^*$ be a dual Banach space, and let T and S be commuting power-bounded operators on X which are duals of operators on Y. Then the following are equivalent for $x\in X$:*

(i) $\sup_{n,m\ge 1}\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1}S^jT^\ell x\|<\infty$.

(ii) x is a double coboundary, i.e. there exists $y\in X$ such that $x=(I-S)(I-T)y$.

(iii) $\sup_{n\ge 1}\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1}S^jT^\ell x\|<\infty$.

Proof. Since (ii) clearly implies (i) and (i) obviously implies (iii), we need to prove only that (iii) implies (ii).

By the proof of (3.1) in Theorem 3.1, (iii) implies that

$$\|\frac{1}{n}\sum_{j=1}^n S^jx+\frac{1}{n}\sum_{\ell=1}^n T^\ell x\|\to 0. \tag{3.6}$$

In order to avoid difficulties with the weak-* topology when Y is not separable, we use the following approach. Let LIM be a fixed Banach limit (defined on ℓ_∞) which extends Cesàro convergence [19, pp. 33-34] (see also

[33, p. 135]). By (iii), we can define

$$\langle y, v\rangle := \text{LIM}\{\langle \sum_{0\le j,\ell\le n-1} S^jT^\ell x, v\rangle\} \quad v\in Y.$$

Then y is a bounded linear functional on Y, i.e. $y \in X$.

Let $\hat{T}$ and $\hat{S}$ be the preduals on Y of T and S. Using the definition, commutativity, (3.6) and the property of LIM preserving Cesàro convergence, for each $v \in Y$ we have

$$\langle (I-T)(I-S)y, v\rangle = \langle y, (I-\hat{T})(I-\hat{S})v\rangle =$$

$$\text{LIM}\{\langle \sum_{0\le j,\ell\le n-1} S^jT^\ell x, (I-\hat{T})(I-\hat{S})v\rangle\} = \text{LIM}\{\langle (I-T^n)(I-S^n)x, v\rangle\} =$$

$$\text{LIM}\{\langle (x-(T^n+S^n)x + (TS)^n x, v\rangle\} = \langle x, v\rangle + \text{LIM}\{\langle (TS)^n x, v\rangle\}.$$

Defining $z \in X$ by $\langle z, v\rangle := \text{LIM}\{\langle (TS)^n x, v\rangle\}$ we obtain $(I-T)(I-S)y = x + z$. By shift-invariance of Banach limits, $(TS)z = z$. We now define $\langle u, v\rangle := \text{LIM}\{\langle (TS)^n y, v\rangle\}$. Then

$$\langle (I-T)(I-S)u, v\rangle = \text{LIM}\{\langle (TS)^n (I-T)(I-S)y, v\rangle\} =$$

$$\text{LIM}\{\langle (TS)^n (x+z, v\rangle = \text{LIM}\{(TS)^n x, v\rangle\} + \langle z, v\rangle = 2\langle z, v\rangle.$$

Hence $2z = (I-T)(I-S)u$, so $x = (I-T)(I-S)(y - \frac{1}{2}u)$. □

Remark. Theorem 3.1 is a corollary of Theorem 3.6. The proof of Theorem 3.1 is closer in spirit to Browder's, and adapts directly to prove Theorem 3.6 when Y is separable.

Corollary 3.7. *Let $X = Y^*$ be a dual Banach space, and let T be a power-bounded operator on X which is the dual of an operator on Y. Then the following are equivalent for $x \in X$:*

(i) $\sup_{n\ge1} \|(\sum_{j=0}^{n-1} T^j)^2 x\| < \infty$.

(ii) There exists $y \in X$ such that $x = (I-T)^2 y$.

Example. *Bounded double cobundaries of commuting Markov operators on L_∞.*

Let P and Q be commuting Markov operators on $(\mathbb{S}, \Sigma)$, and let μ be a finite measure such that the measures μP and μQ, defined by $(\mu P)(A) := \int P(s, A)d\mu(s)$ and $(\mu Q)(A) := \int Q(s, A)d\mu(s)$, are absolutely continuous with respect to μ. Then the operators $\hat{T}$ and $\hat{S}$, defined on the space $M(\mathbb{S}, \Sigma, \mu)$ of finite signed measures absolutely continuous with respect to μ by $\hat{T}\nu(A) = \int P(s, A)d\nu(s)$ and $\hat{S}\nu(A) = \int Q(s, A)d\nu(s)$, satisfy $\hat{T}^* = P$ and $\hat{S}^* = Q$ on $M(\mathbb{S}, \Sigma, \mu)^* = L_\infty(\mathbb{S}, \Sigma, \mu)$. Theorem 3.6 yields that the following are equivalent for $f \in L_\infty(\mathbb{S}, \Sigma, \mu)$:

(i) $\sup_{n,m\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} P^jQ^\ell f\|_\infty < \infty$.

(ii) f is a double coboundary for P and Q.

(iii) $\sup_{n\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} P^jQ^\ell f\|_\infty < \infty$.

Remarks. 1. The separability of $L_1(\mathbb{S}, \Sigma, \mu)$ (identified with $M(\mathbb{S}, \Sigma, \mu)$ via the Radon-Nikodým theorem), used in the proof of Proposition 3.5, is not needed.

2. The previous example applies to the characterization of bounded double coboundaries of commuting non-singular transformations.

Corollary 3.8. *Let T and S be commuting power-bounded operators on a Banach space X. If* $\sup_{n\ge1}\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1}S^jT^\ell x\|<\infty$, *then there exists* $\phi\in X^{**}$ *such that*

$$x=(I-T^{**})(I-S^{**})\phi,$$

and then $\sup_{n,m\ge1}\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1}S^jT^\ell x\|<\infty$.

Proof. We identify X with its canonical embedding in X^{**}, and then T and S are the restrictions to X of T^{**} and S^{**}. Now apply Theorem 3.6 to T^{**} and S^{**}. □

Remark. By corollary (3.8), if $\sup_{n\ge1}\|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1}S^jT^\ell x\|<\infty$, then $\|\frac{1}{n}\sum_{\ell=1}^{n}T^\ell x\|\to0$ and $\|\frac{1}{n}\sum_{j=1}^{n}S^j x\|\to0$, which is a strengthening of (3.1).

As an application, we look at commuting irreducible Markov operators on compact spaces. Let K be a compact Hausdorff space. A *Markov operator* on $C(K)$ is a positive linear operator P on $C(K)$ with $P\mathbf{1}=\mathbf{1}$; it is given by the transition probability $P(s,A)=P^*\delta_s(A)$. A Markov operator is called *irreducible* if the only absorbing closed set is K, and *uniquely ergodic* if it has only one invariant probability μ (a probability μ is called *invariant* if $\mu P:=P^*\mu=\mu$; an invariant probability always exists [30, p. 178]). A uniquely ergodic Markov operator is irreducible if and only if the support of the (unique) invariant probability is K (see [31], [30, pp. 177-179] for more details).

If μ is an invariant probability for P, then P defines a Markov operator on $L_\infty(\mu)$; when P on $L_\infty(\mu)$ is ergodic ($Pf=f\in L_\infty(\mu)$ holds only if f is constant a.e.), μ is called *ergodic.* Since the set of P-invariant probabilities on K is non-empty, convex and weak-* compact, by the Krein-Milman theorem [17, p. 440] it has extreme points, which are precisely the ergodic probabilities for P.

Lemma 3.9. *Let K be a compact Hausdorff space and P an irreducible Markov operator on $C(K)$. Let $f\in C(K)$ and assume that for some P-invariant probability μ there is $\psi\in L_\infty(\mu)$ such that $f=(I-P)\psi$ a.e. Then there exists $g\in C(K)$ with $f=(I-P)g$.*

Proof. By assumption, $|\sum_{k=0}^{n}P^kf(s)|=|(I-P^{n+1})\psi(s)|\le2\|\psi\|_{L_\infty(\mu)}$ a.e.

Define $A:=\{s\in K:\sup_n|\sum_{k=0}^{n}P^kf(s)|>3\|\psi\|_{L_\infty(\mu)}\}$. Then A is open, and by assumption $\mu(A)=0$. Since the support of μ is K, by irreducibility, $A=\emptyset$. Hence $\sup_n\|\sum_{k=0}^{n}P^kf\|_{C(K)}<\infty$, and by [31] there exists $g\in C(K)$ with $f=(I-P)g$. □

Remarks. 1. Without irreducibility the result of [31] may fail. Example 3 in [37] exhibits P (induced by a continuous map) uniquely ergodic but not irreducible on $C(K)$ of a compact metric space, and $f\notin(I-P)C(K)$ with $\sup_n\|\sum_{k=0}^{n}P^kf\|_{C(K)}<\infty$.

2. For P induced by a minimal continuous map of K, Lemma 3.9 is a special case of Theorem 1 of Quas [40].

Theorem 3.10. *Let P and Q be commuting irreducible Markov operators on $C(K)$ of a compact Hausdorff space K, and assume they have a common* *invariant probability μ which is ergodic for P. If $f \in C(K)$ satisfies*

$$\sup_{n\geq 1} \| \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} P^j Q^\ell f \|_{L_\infty(\mu)} < \infty,$$

then there exists a function $g \in C(K)$ such that $f = (I-P)(I-Q)g$.

Proof. By invariance of μ, P and Q are contractions also of $L_\infty(\mu)$, and are the duals of the contractions induced on $L_1(\mu)$, via the Radon-Nikodým theorem, by P^* and Q^* which preserve absolute continuity with respect to μ. Hence by Theorem 3.6 there exists $\psi \in L_\infty(\mu)$ such that $f = (I-P)(I-Q)\psi$ a.e.-μ. We apply Lemma 3.9 to P and obtain a function $h \in C(K)$ with $f = (I-P)h$. Since μ is ergodic, when we normalize h to have $\int h\, d\mu = 0$ we obtain that $h = (I-Q)\psi$ a.e.-μ. We now apply Lemma 3.9 to Q, and obtain a function $g \in C(K)$ with $h = (I-Q)g$. Hence $f = (I-P)h = (I-P)(I-Q)g$. □

Remark. The set of probabilities on K is convex and weak-* compact; since it is invariant under P^* and Q^*, by the Markov-Kakutani fixed point theorem [17, p. 456] it contains a common invariant probability. What the proof of Theorem 3.10 needs is a common invariant probability which is *ergodic* for at least one of the operators.

Corollary 3.11. *let P be an irreducible Markov operator on $C(K)$ of a compact Hausdorff space K. Then $f \in (I-P)^2 C(K)$ if and only if*

$$\sup_{n\geq 1} \| (\sum_{j=0}^{n-1} P^j)^2 f \|_{C(K)} < \infty,$$

Corollary 3.12. *Let P and Q be commuting irreducible uniquely ergodic Markov operators on $C(K)$ of a compact Hausdorff space K. Then the following are equivalent for $f \in C(K)$:*

(i) $\sup_{n,m\geq 1} \| \sum_{j=0}^{n-1} \sum_{\ell=0}^{m-1} P^j Q^\ell f \|_{C(K)} < \infty$.

(ii) f is a double coboundary for P and Q.

(iii) $\sup_{n\geq 1} \| \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} P^j Q^\ell f \|_{C(K)} < \infty$.

Proof. We have only to show that (iii) implies (ii). First note that P and Q have the same invariant probability: if μ is the unique invariant probability of P, then $(\mu Q)P = (\mu P)Q = \mu Q$, and by uniqueness $\mu Q = \mu$. The invariant probability μ is ergodic by unique ergodicity, and is supported by K by irreducibility of P and Q. We can now apply the previous theorem and obtain (ii). □

Example. *Continuous double coboundaries of convolutions on the circle.* Let ν and η be two probabilities on the unit circle, whose supports contain an "irrational" point. Hence they are uniquely ergodic and commute, so the corollary can be applied. A particular case is when ν and η are irrational rotations.

4. Double coboundaries of commuting contractions in L_1

Although L_1 is not reflexive and contractions need not be duals, Browder's theorem was extended to *contractions* of L_1 in [37]. The following theorem, based on this result, yields as corollary the case $p = 1$ of Proposition 3.5.

Theorem 4.1. *Let $(\mathbb{S}, \Sigma, \mu)$ be a σ-finite measure space, let T and S be be commuting contractions on $L_1(\mathbb{S}, \Sigma, \mu)$, and assume that T is mean ergodic. Then the following are equivalent for $f \in L_1(\mu)$:*
(i) $\sup_{n,m\geq 1} \| \sum_{j=0}^{n-1} \sum_{\ell=0}^{m-1} S^j T^\ell f\| < \infty$.
(ii) f is a double coboundary of T and S.
(iii) $\sup_{n\geq 1} \| \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} S^j T^\ell f\| < \infty$.

Proof. We have only to prove that (iii) implies (ii).
By Corollary 3.8 also (i) holds, which implies that

$$\sup_n \| \sum_{\ell=0}^{n-1} T^\ell f\| = \sup_n \| \sum_{\ell=0}^{n-1} T^{**\ell} f\| < \infty.$$

Hence, by [37], there exists $h \in L_1(\mu)$ with $(I-T)h = f$. Since T is assumed mean ergodic, we may assume that $\lim \|\frac{1}{n}\sum_{\ell=1}^{n} T^\ell h\| \to 0$, and then

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{\ell=0}^{n-1} T^\ell f = \frac{1}{N}\sum_{n=1}^{N}\sum_{\ell=0}^{n-1} T^\ell (I-T)h = \frac{1}{N}\sum_{n=1}^{N}(I-T^n)h \underset{N\to\infty}{\to} h, \tag{4.1}$$

with convergence in norm. Let M be the supremum in (i). Then

$$\sup_{N,m\geq 1} \|\frac{1}{N}\sum_{n=1}^{N}\sum_{\ell=0}^{n-1} T^\ell \sum_{j=0}^{m-1} S^j f\| \leq M. \tag{4.2}$$

For fixed m this yields

$$\| \sum_{j=0}^{m-1} S^j h\| = \| \sum_{j=0}^{m-1} S^j (\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\sum_{\ell=0}^{n-1} T^\ell f)\| =$$

$$\lim_N \| \sum_{j=0}^{m-1} S^j \frac{1}{N}\sum_{n=1}^{N}\sum_{\ell=0}^{n-1} T^\ell f\| \leq M.$$

We now apply again [37]: there exists $g \in L_1(\mu)$ such that $(I-S)g = h$. Hence $f = (I-T)h = (I-T)(I-S)g$. □

Remark. We conjecture that the theorem is true without assuming mean ergodicity of one of the contractions, but we have not been able to prove it. The mean ergodicity of T was used in obtaining the iterative solution (4.1) of Poisson's equation. Without mean ergodicity, the left-hand side of (4.1) need not converge even weakly to a solution, although there is one (see [37, Example 1]).

Definition. A contraction T on L_1 is said to *satisfy the pointwise ergodic theorem* if for every $f \in L_1$ the ergodic averages converge a.e. to a (necessarily integrable, by Fatou's lemma) T-invariant function.

Lemma 4.2. *Let T be a positive contraction on L_1. Let $0 \le f \in L_1$ have a.e. convergent ergodic averages. Then the limit is an integrable invariant function.*

Proof. Put $g := \lim_n \frac{1}{n}\sum_{k=1}^n T^k f$. By Fatou's lemma g is integrable, and

$$Tg = T(\lim_n \frac{1}{n}\sum_{k=1}^n T^k f) \le \liminf T(\frac{1}{n}\sum_{k=1}^n T^k f) = g,$$

since $\liminf \frac{1}{n}T^{n+1}f = 0$ a.e. by Fatou's lemma. Thus $Tg \le g$, and by [33, Lemma 3.10, p. 131] $Tg = g$ on the conservative part C. But by Hopf's decomposition $\sum_{k=1}^\infty T^k f < \infty$ a.e. on the dissipative part D, so $g = 0$ on D, and $0 \le Tg \le g$ shows that $Tg = g$ on D. Hence $Tg = g$. □

A positive mean ergodic contraction of L_1 satisfies the pointwise ergodic theorem [26], but in general, if T is a mean ergodic contraction and its linear modulus is *not* mean ergodic, T need not satisfy the pointwise ergodic theorem [13, p. 115].

Theorem 4.3. *Let $(\mathbb{S}, \Sigma, \mu)$ be a σ-finite measure space,and let T and S be be commuting contractions on $L_1(\mathbb{S}, \Sigma, \mu)$. Assume that T satisfies the pointwise ergodic theorem, and S preserves almost everywhere convergence of sequences of integrable functions. Then the following are equivalent for $f \in L_1(\mu)$:*

(i) $\sup_{n,m\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} S^jT^\ell f\| < \infty$.

(ii) f is a double coboundary of T and S.

(iii) $\sup_{n\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^jT^\ell f\| < \infty$.

Proof. The proof follows the proof of Theorem 4.1, till (4.1), but now in (4.1) the convergence to the solution h of Poisson's equation is almost everywhere; Lemma 4.2 is used for the assumption that $\frac{1}{n}\sum_{\ell=1}^n T^\ell h \to 0$ a.e.

Put $h_N := \frac{1}{N}\sum_{n=1}^N \sum_{k=0}^{n-1} T^k f$. Then $h_N \in L_1$ and converges a.e. to h. Since S preserves a.e. convergence of sequences of integrable functions, so do its powers. For fixed m we use Fatou's lemma and (4.2) to obtain

$$\|\sum_{j=0}^{m-1} S^j h\| = \|\sum_{j=0}^{m-1} S^j(\lim_{N\to\infty} h_N)\| = \|\lim_N \sum_{j=0}^{m-1} S^j h_N\| \le$$

$$\liminf_{N\to\infty} \|\sum_{j=0}^{m-1} S^j h_N\| \le M.$$

We now apply again [37]: there exists $g \in L_1(\mu)$ such that $(I-S)g = h$. Hence $f = (I-T)h = (I-T)(I-S)g$. □

Corollary 4.4. *Let θ and τ be commuting measure preserving transformations on a σ-finite measure space $(\mathbb{S}, \Sigma, \mu)$, and let T and S be the contractions they induce on $L_1(\mathbb{S}, \Sigma, \mu)$. Then the following are equivalent for $f \in L_1(\mu)$:*

(i) $\sup_{n,m\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} S^jT^\ell f\| < \infty$.

(ii) f is a double coboundary of T and S.

(iii) $\sup_{n\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^jT^\ell f\| < \infty$.

Example. *Contractions to which Theorem 4.3 applies but Theorem 4.1 does not.*

Let θ and τ be commuting measure preserving transformations on a σ-finite measure space $(\mathbb{S}, \Sigma, \mu)$, and let T and S be the contractions they induce on $L_1(\mathbb{S}, \Sigma, \mu)$. Let $T_1 := \sum_{j\geq 0} a_j T^j$, with $a_j \geq 0$ and $\sum_{j\geq 0} a_j = 1$. Then T_1 is a Markov operator having μ as invariant measure. By the Hopf-Dunford-Schwartz theorem, T_1 satisfies the pointwise ergodic theorem, and commutes with S since T does. Corollary 4.4 does not apply to T_1 and S, but Theorem 4.3 does. When μ is infinite and the transformations are ergodic, Theorem 4.1 does not apply.

Example. *Contractions to which Theorem 4.1 applies but Theorem 4.3 does not.*

Let ν and η be continuous probabilities on the unit circle $\mathbb{T}$ with all convolution powers singular, and denote by λ the normalized Lebesgue measure. On $L_1(\lambda)$ define $Tf := \nu * f$ and $Sf := \eta * f$. Then T and S are commuting Markov operators which preserve λ. Hence they are mean ergodic on $L_1(\lambda)$, so Theorem 4.1 applies. Theorem 4.3 need not apply. The assumption of singular convolution powers is since if some power of ν (or η) has an absolutely continuous component, then T (or S) is uniformly ergodic [4].

Our next result uses Komlós's theorem [28]. Aaronson and Weiss [1] suggested to use Komlós's theorem for solving Poisson's equation (the cohomology equation) in L_1 for a single probability preserving transformation (a special case of [37]). We note that Komlós's theorem, although stated in probabilistic notation, is valid in σ-finite measure spaces, since we can always change the measure to an equivalent probability, and obtain an isomorphism of the spaces of integrable functions which preserves pointwise convergence.

Theorem 4.5. *Let $(\mathbb{S}, \Sigma, \mu)$ be a σ-finite measure space and let T and S be commuting contractions on $L_1(\mathbb{S}, \Sigma, \mu)$. Assume that T and S preserve almost everywhere convergence of sequences of integrable functions. Then the following are equivalent for $f \in L_1(\mu)$:*

(i) $\sup_{n,m\geq 1} \| \sum_{j=0}^{n-1} \sum_{\ell=0}^{m-1} S^j T^\ell f \| < \infty$.

(ii) f is a double coboundary of T and S.

(iii) $\sup_{n\geq 1} \| \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} S^j T^\ell f \| < \infty$.

Proof. The proof follows the proof of Theorem 4.1, till the identity part of (4.1).

Without loss of generality μ is a probability. Since $\|T^{n+1}h\|/n \to 0$, it converges in probability, hence there is a subsequence with $T^{N_k+1}h/N_k \to 0$ a.e. The sequence $\{\frac{1}{N_k}\sum_{n=1}^{N_k} T^n h\}$ is norm bounded in L_1, so by Komlós's theorem there exist $h' \in L_1$ and a subsequence $\{k_i\}$ such that

$$\frac{1}{K}\sum_{i=1}^{K}\Big(\frac{1}{N_{k_i}}\sum_{n=1}^{N_{k_i}} T^n h\Big) \underset{K\to\infty}{\to} h' \quad a.e.$$

Since T preserves a.e. convergence of sequences of integrable functions,

$$Th' = \lim_{K\to\infty} \frac{1}{K}\sum_{i=1}^{K}\Big(\frac{1}{N_{k_i}}\sum_{n=1}^{N_{k_i}} T^{n+1}h\Big) =$$

$$h' + \lim_{K\to\infty} \frac{1}{K}\sum_{i=1}^{K}\frac{1}{N_{k_i}}(T^{N_{k_i}+1}h - Th) = h'.$$

We then obtain from (4.1) that

$$\frac{1}{K}\sum_{i=1}^{K}\frac{1}{N_{k_i}}\sum_{n=1}^{N_{k_i}}\sum_{\ell=0}^{n-1} T^\ell f = \frac{1}{K}\sum_{i=1}^{K}\frac{1}{N_{k_i}}\sum_{n=1}^{N_{k_i}}(I-T^n)h \underset{K\to\infty}{\to} h - h' \quad a.e.$$

Since h' is T-invariant (zero if T has no fixed points), also $(I-T)(h-h') = f$, so replacing h by $h-h'$ we may assume $h'=0$, and then

$$h_K := \frac{1}{K}\sum_{i=1}^{K}\frac{1}{N_{k_i}}\sum_{n=1}^{N_{k_i}}\sum_{\ell=0}^{n-1} T^\ell f \to h \quad a.e.$$

By averaging in (4.2) we obtain

$$\sup_{K,m\geq 1} \|\sum_{j=0}^{m-1} S^j h_K\| = \sup_{K,m\geq 1} \|\frac{1}{K}\sum_{i=1}^{K}\frac{1}{N_{k_i}}\sum_{n=1}^{N_{k_i}}\sum_{\ell=0}^{n-1} T^\ell \sum_{j=0}^{m-1} S^j f\| \leq M. \tag{4.3}$$

Since S preserves a.e. convergence of sequences of integrable functions, so do its powers. For fixed $m \geq 1$, Fatou's lemma and (4.3) yield

$$\|\sum_{j=0}^{m-1} S^j h\| = \|\sum_{j=0}^{m-1} S^j(\lim_{K\to\infty} h_K)\| = \|\lim_K \sum_{j=0}^{m-1} S^j h_K\| \leq$$

$$\liminf_{K\to\infty} \|\sum_{j=0}^{m-1} S^j h_K\| \leq M.$$

By [37], there exists $g \in L_1(\mu)$ such that $(I-S)g = h$. Hence $f = (I-T)h = (I-T)(I-S)g$. □

Example. *Contractions to which Theorem 4.5 applies, Theorems 4.1 and 4.3 do not.*

Let τ be the non-singular invertible transformation on $[0,1]$ with Lebesgue measure μ, constructed by Chacon (see [33, pp. 151-153]); its pre-dual operator $Tf(x) := \frac{d\mu\circ\tau^{-1}}{d\mu}(x)f(\tau^{-1}x)$ induced on L_1 (with $T^*g = g\circ\tau$) does not satisfy the pointwise ergodic theorem, and since T is positive, it is not mean ergodic [26]. However, its structure shows that it preserves a.e. convergence of sequences of integrable functions. Looking at $\tau\times\tau$ on the unit square, we obtain two commuting contractions T_1 and T_2 which are not mean ergodic and do not satisfy the pointwise ergodic theorem, but each preserves pointwise convergence of sequences of integrable functions.

Remarks. 1. Theorem 4.5 does not apply in the examples following Corollary 4.4, so Theorems 4.1, 4.3 and 4.5 are not comparable.

2. Corollary 4.4 is also a consequence of Theorem 4.5.

We now present a different proof of Theorem 4.5, which may be of interest. It uses (3.1), but does not use Corollary 3.8, nor [37]. Instead, the strong Komlós theorem [28, Theorem 1a] is used.

Proof. We have only to show that (iii) implies (ii). Define

$$g_n := \frac{1}{n}\sum_{k=1}^{n}\sum_{0\le j,\ell\le k-1} S^jT^\ell f = \frac{1}{n}\sum_{k=1}^{n}(\sum_{j=0}^{k-1} S^j)(\sum_{\ell=0}^{k-1} T^\ell)f.$$

By (iii), $\sup_n \|g_n\|_1 < \infty$. Also the sequences $\{\frac{1}{n}\sum_{k=1}^{n}(T^kf + S^kf)\}$, and $\{\frac{1}{n}\sum_{k=1}^{n}(TS)^kf\}$ are norm bounded in L_1.

By applying the strong version of Komlós's theorem [28, Theorem 1a] successively three times, there exist an increasing subsequence of integers $\{m_r\}$ and functions $g, h_1 \in L_1$ such that for every subsequence $\{n_r\} \subset \{m_r\}$,

$$\frac{1}{n}\sum_{r=1}^{n} g_{n_r} \to g \quad a.e., \qquad \frac{1}{n}\sum_{r=1}^{n}\frac{1}{n_r}\sum_{k=1}^{n_r}(TS)^kf \to h_1 \quad a.e.,$$

and $\frac{1}{n}\sum_{r=1}^{n}\frac{1}{n_r}\sum_{k=1}^{n_r}(T^k + S^k)f$ converges a.e. By the proof of (3.1) in Theorem 3.1, (iii) implies that $\|\frac{1}{n}\sum_{j=1}^{n} S^jf + \frac{1}{n}\sum_{\ell=1}^{n} T^\ell f\|_1 \to 0$, so $\frac{1}{n}\sum_{r=1}^{n}\frac{1}{n_r}\sum_{k=1}^{n_r}(T^k + S^k)f$ converges a.e. to 0, by Fatou's lemma.

By assumption, T and S, and therefore also TS, preserve almost everywhere convergence. With the limits in the following equations being pointwise a.e. limits, we obtain

$$(I-T)(I-S)g = \lim_n \frac{1}{n}\sum_{r=1}^{n}(I-T)(I-S)g_{m_r} =$$

$$\lim_n \frac{1}{n}\sum_{r=1}^{n}\frac{1}{m_r}\sum_{k=1}^{m_r}(I-T)(I-S)(\sum_{j=0}^{k-1} S^j)(\sum_{\ell=0}^{k-1} T^\ell)f =$$

$$\lim_n \frac{1}{n}\sum_{r=1}^{n}\frac{1}{m_r}\sum_{k=1}^{m_r}(I-T^k)(I-S^k)f =$$

$$\lim_n \frac{1}{n}\sum_{r=1}^{n}\frac{1}{m_r}\sum_{k=1}^{m_r}(f - T^kf - S^kf + (TS)^kf).$$

We thus have $(I-T)(I-S)g = f + h_1$. We now use Komlós's theorem to obtain a function $h_2 \in L_1$ and a subsequence $\{n_r\} \subset \{m_r\}$ such that $\frac{1}{N}\sum_{r=1}^{N}\frac{1}{n_r}\sum_{k=1}^{n_r}(TS)^kg$ converges a.e. to h_2. Since $(TS)h_1 = h_1$, we obtain

$$(I-T)(I-S)h_2 = \lim_N \frac{1}{N}\sum_{r=1}^{N}\frac{1}{n_r}\sum_{k=1}^{n_r}(TS)^k(I-T)(I-S)g =$$

$$\lim_N \frac{1}{N}\sum_{r=1}^{N}\frac{1}{n_r}\sum_{k=1}^{n_r}(TS)^k(f+h_1) = 2h_1.$$

Hence $f = (I-T)(I-S)g - h_1 = (I-T)(I-S)(g - \frac{1}{2}h_2)$. □

Notations. We denote by $M(\mathbb{S},\Sigma,\mu)$ the space of countably additive finite signed measures absolutely continuous with respect to μ, and by $ba(\mathbb{S},\Sigma,\mu)$ the space of bounded finitely additive measures (charges) vanishing on the null sets of μ. It is known that $ba(\mathbb{S},\Sigma,\mu)$ is the second dual of $M(\mathbb{S},\Sigma,\mu)$ [17, p. 296]. By the Yosida-Hewitt decomposition [45, Theorem 1.24], every $\nu \in ba(\mathbb{S},\Sigma,\mu)$ can be *uniquely* decomposed as $\nu = \nu_1 + \nu_0$, with $\nu_1 \in M(\mathbb{S},\Sigma,\mu)$ countably additive and $\nu_0 \in ba(\mathbb{S},\Sigma,\mu)$ a pure charge (i.e. $|\nu_0|$ does not bound any countably additive non-negative measure). S. Horowitz [25] showed that $L_\infty(\mathbb{S},\Sigma,\mu)$ is isometrically and order isomorphic to $C(K)$ of a compact Hausdorff space in a way that $M(\mathbb{S},\Sigma,\mu)$ is isometrically order isomorphic to the finite signed measures absolutely continuous with respect to a probability $\hat{\mu}$, and pure charges in $L_\infty(\mu)$ correspond to the finite signed measures on K singular to $\hat{\mu}$.

For our next result we need the following lemma.

Lemma 4.6. *Let T be a contraction on $M(\mathbb{S},\Sigma,\mu)$ and let $\eta \in ba(\mathbb{S},\Sigma,\mu)$ satisfy $T^{**}\eta = \eta$. Then η_c, the countably additive part of η, satisfies $T\eta_c = \eta_c$.*

Proof. Let $\eta_p = \eta - \eta_c$ be the pure charge part of η. Then (as in [37])

$$\|\eta_p\| \geq \|T^{**}\eta_p\| = \|\eta_c + \eta_p - T\eta_c\| = \|\eta_p\| + \|\eta_c - T\eta_c\|,$$

since η_c and η_p are "mutually singular". Hence $T\eta_c = \eta_c$. □

The authors are grateful to Christophe Cuny for the idea of the proof of the next theorem.

Theorem 4.7. *Let $(\mathbb{S},\Sigma,\mu)$ be a σ-finite measure space, and let T and S be commuting contractions on $L_1(\mathbb{S},\Sigma,\mu)$. If T has no non-zero invariant functions, or if S is an invertible isometry, then the following are equivalent for $f \in L_1$:*

(i) $\sup_{n,m\geq 1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} S^jT^\ell f\| < \infty$.

(ii) f is a double coboundary of T and S.

(iii) $\sup_{n\geq 1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^jT^\ell f\| < \infty$.

Proof. We have to prove only that (iii) implies (ii).

Via the Radon-Nikodým theorem, we identify $L_1(\mathbb{S},\Sigma,\mu)$ with $M(\mathbb{S},\Sigma,\mu)$. Then ν, defined by $d\nu = fd\mu$, satisfies $\sup_{n\geq 1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^jT^\ell \nu\| < \infty$.

By Corollary 3.8, $\nu = (I-T^{**})(I-S^{**})\psi$ for some $\psi \in ba(\mathbb{S},\Sigma,\mu)$. Put $\eta := (I-S^{**})\psi$. Then $\sup_n \|\sum_{k=0}^{n-1} S^{**k}\eta\| = M < \infty$. Decompose $\eta = \eta_1 + \eta_0$ with η_1 countably additive and η_0 a pure charge.

Since $(I-T^{**})\eta = \nu$, the proof of [37] shows that $(I-T)\eta_1 = \nu$ and $T^{**}\eta_0 = \eta_0$. Since S^{**} preserves countable additivity, $\sum_{k=0}^{n-1} S^{**k}\eta_1$ is countably additive (and equals $\sum_{k=0}^{n-1} S^k\eta_1$). To finish the proof we have to show that these sums are norm-bounded.

Case 1: T has no non-zero fixed points in $M(\mathbb{S},\Sigma,\mu)$.
Since T and S commute, $T^{**}(S^{**k}\eta_0) = S^{**k}\eta_0$. Hence by Lemma 4.6 and the assumption, $\sum_{k=0}^{n-1} S^{**k}\eta_0$ is a pure charge for every n. Hence

$$\|\sum_{k=0}^{n-1} S^k\eta_1\| \leq \|\sum_{k=0}^{n-1} S^{**k}\eta_1\| + \|\sum_{k=0}^{n-1} S^{**k}\eta_0\| = \|\sum_{k=0}^{n-1} S^{**k}\eta\| \leq M. \tag{4.4}$$

Case 2: S is an invertible isometry.
By [10, Lemma 4.2], $S^{**k}\eta_0$ is a pure charge for any k, so as before (4.4) holds.

In either case, we obtain $\sup_n \|\sum_{k=0}^{n-1} S^k\eta_1\| \le M$, so by [37] there is a $\zeta \in M(\mathbb{S},\Sigma,\mu)$ with $(I-S)\zeta = \eta_1$; hence $(I-T)(I-S)\zeta = (I-T)\eta_1 = \nu$. □

Definition. Let T be a positive contraction of $L_1(\mathbb{S},\Sigma,\mu)$. A set $B \in \Sigma$ is called T-*absorbing* if $Tf \in L_1(B)$ whenever $f \in L_1(B)$. This is equivalent to $T^*1_{\mathbb{S}-B} \le 1_{\mathbb{S}-B}$ [33, p. 118]. T is called *irreducible* if the only non-null T-absorbing set is $\mathbb{S}$ (mod μ).

Corollary 4.8. *Let $(\mathbb{S},\Sigma,\mu)$ be a σ-finite measure space, and let T and S be commuting contractions on $L_1(\mathbb{S},\Sigma,\mu)$. If T is an irreducible positive contraction, then the following are equivalent for $f \in L_1$:*
(i) $\sup_{n,m\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} S^jT^\ell f\| < \infty$.
(ii) f is a double coboundary of T and S.
(iii) $\sup_{n\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^jT^\ell f\| < \infty$.

Proof. The set C, the conservative part of T, is T-absorbing [33, p. 118], so by irreducibility either T is dissipative or T is conservative. If T is dissipative, it has no non-zero fixed point (all invariant probabilities are supported by C [33, p. 141]), and Theorem 4.7 applies.

We now assume that T is conservative, so it is ergodic by irreducibility. If T has a fixed point $h \ne 0$ in L_1, then, by irreducibility, $|h|/\|h\|$ defines an *equivalent* invariant probability [33, p. 132], and then T is mean ergodic (e.g. [33, p. 73]), so Theorem 4.1 applies. If T has no fixed points in L_1 (except 0), then Theorem 4.7 applies. □

Remarks. 1. Theorem 4.7 applies when T is a dissipative positive contraction, even without irreducibility, since it has no fixed points, as observed in the previous proof.

2. A conservative and ergodic positive contraction T on L_1 is irreducible, since T-absorbing sets are invariant.

Theorem 4.9. *Let $(\mathbb{S},\Sigma,\mu)$ be a σ-finite measure space, and let T and S be commuting conservative positive contractions on $L_1(\mathbb{S},\Sigma,\mu)$. Then the following are equivalent for $f \in L_1$:*
(i) $\sup_{n,m\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{m-1} S^jT^\ell f\| < \infty$.
(ii) f is a double coboundary of T and S.
(iii) $\sup_{n\ge1} \|\sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S^jT^\ell f\| < \infty$.

Proof. Since T is conservative, there exists a decomposition of the space $\mathbb{S} = C_1 \cup C_0$, with each C_i invariant (i.e. $T * 1_{C_i} = 1_{C_i}$ for $i = 0,1$), such that every fixed point of T vanishes on C_0, while there exists $0 \le p \in L_1(\mathbb{S})$ with $Tp = p$ and $\{p > 0\} = C_1$ [33, p. 141]. If $\mu(C_0) = 0$ or $\mu(C_1) = 0$, the desired equivalence is proved like Corollary 4.8 (and S needs to be only a contraction of $L_1(\mathbb{S})$, not necessarily positive). So we assume now that $\mu(C_0) > 0$ and $\mu(C_1) > 0$.

By commutation, $T(Sp) = S(Tp) = Sp$, so the property of C_0 yields $\{Sp > 0\} \subset C_1$. It follows easily that if $h \in L_1(C_1)$ (i.e. $h \in L_1(\mathbb{S})$

supported on C_1), then Sh is also supported on C_1. Hence $L_1(C_1)$ is invariant under S (i.e. C_1 is absorbing for S), which yields $S^*1_{C_0} \le 1_{C_0}$. Since S is conservative, $S^*1_{C_0} = 1_{C_0}$, and also $S^*1 = 1$, which implies that $S^*1_{C_1} = 1_{C_1}$. This shows that also C_0 is absorbing for S, and we conclude that $L_1(C_0)$ and $L_1(C_1)$ are both invariant under T and under S. Denote $T_i = T_{|L_1(C_i)}$ and $S_i = S_{|L_1(C_i)}$.

Let $f \in L_1(\mathbb{S})$ satisfy (iii). Writing $f = f_1 + f_0$ with $f_i = f1_{C_i}$, we obtain

$$\sup_{n\ge 1} \Big[\| \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S_0^j T_0^\ell f_0\| + \| \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} S_1^j T_1^\ell f_1\|\Big] < \infty.$$

Since T_1 is mean ergodic and T_0 has no fixed points (except 0), Theorems 4.1 and 4.7 yield the existence of $g_i \in L_1(C_i)$ such that $(I - T_i)(I - S_i)g_i = f_i$, $i = 0, 1$. Hence (ii) holds with $g = g_1 + g_0$. □

Example. *Contractions to which Theorem 4.7 applies, Theorems 4.1, 4.3 and 4.5 do not.*

Let ν and η be absolutely continuous probabilities on $\mathbb{R}$, and define on $L_1(\mathbb{R})$ the convolution operators $Tf = \nu * f$ and $Sf = \eta * f$. Neither T nor S has fixed points, so Theorem 4.7 applies. Neither operator is mean ergodic in L_1. Both satisfy the pointwise ergodic theorem, but they do not preserve a.e. convergence of sequences of integrable functions. T and S are irreducible, so also Corollary 4.8 applies.

Remark. In the second example following Corollary 4.4, also Theorem 4.7 does not apply. However, Theorem 4.9 does apply to that example.

Problem. *Is Theorem 4.1 true without the assumption that T (or S) is mean ergodic?*

The problem is whether (iii) in Theorem 4.1 implies (ii) without any additional assumptions on T or S. Theorems 4.3 and 4.5 put different additional assumptions on both T and S, and as remarked above, these three theorems are not comparable. However, they suggest that they might just be special cases of a general result which does not require additional assumptions. An important special case of the problem is whether the three conditions in Theorem 4.1 are equivalent when T and S are both positive contractions. The answer is positive when one of them is dissipative (remark following Corollary 4.8) or when both are conservative (Theorem 4.9), but we do not know the answer in the general case.

Acknowledgement. The authors are grateful to Catalin Badea for providing them with a copy of the thesis [16], which brought [32] to their attention.

References

[1] J. Aaronson and B. Weiss, *Remarks on the tightness of cocycles*, Colloq. Math. **84/85** (2000), Part 2, 363–376.

[2] A. Alonso, J. Hong and R. Obaya, *Absolutely continuous dynamics and real coboundary cocycles in L^p-spaces,* $0 < p < \infty$, Studia Math. **138** (2000), 121–134.

[3] D.V. Anosov, *On an additive functional homology equation connected with an ergodic rotation of the circle*, Math. USSR Izwestia **7** (1973), 1257–1271.

[4] R.N. Bhattacharya, *Speed of convergence of the n-fold convolution of a probability measure on a compact group*, Z. Wahrscheinlichkeitstheorie u, Verw. Geb. **25** (1972), 1–10.

[5] R. C. Bradley, *An L_p "cousin of coboundary" theorem for random fields*, J. Australian Math. Soc. **85** (2008), 51–58.

[6] J. Brémont and Z. Buczolich, *Maximizing points and coboundaries for an irrational rotation on a circle*, Ergodic Th. Dyn. Sys. **33** (2013), 24–48.

[7] F. Browder, *On iteration of transformations in non-compact minimal dynamical systems*, Proc. Amer. Math. Soc. **9** (1958), 773–780.

[8] F. Browder and W. Petryshyn, *The solution by iteration of linear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 566–570.

[9] P. Butzer and U. Westphal, *The mean ergodic theorem and saturation*, Indiana U. Math. J. **20** (1971), 1163–1174.

[10] G. Cohen, C. Cuny and M. Lin, *The one-sided ergodic Hilbert transform in Banach spaces*, Studia Math. **196** (2010), 251–263.

[11] G. Cohen and J.-P. Conze, *CLT for random walks of commuting endomorphisms on compact Abelian groups*, J. Theoretical Probab., to appear.

[12] G. Cohen and J.-P. Conze, *Central limit theorem for commutative semigroups of toral automorphisms*, preprint, Arxiv 1304.4556 (37 pp.).

[13] D. Çömez and M. Lin, *mean ergodicity of L_1 contractions and pointwise ergodic theorems*, in Almost Everywhere Convergence II, Academic Press, Boston, 1991, pp. 113–126.

[14] Y. Derriennic and M. Lin, *The central limit theorem for random walks on orbits of probability preserving transformations*, Contemp. Math. **444** (2007), 31–51.

[15] Y. Derriennic and M. Lin, *Variance bounding Markov chains, L_2-uniform mean ergodicity and the CLT*, Stochastics and Dynamics **11** (2011), 81–94.

[16] O. Devys, *Localisation spectrale à l'aide de polynômes de Faber et équation de cobord*, Ph.D. thesis, Université Lille 1, 2012.

[17] N. Dunford and J. Schwartz, *Linear Operators, part I*, Wiley Interscience, New York, 1958.

[18] M. El Machkouri and D. Giraudo, *Orthomartingale-coboundary decomposition for stationary random fields*, Arxiv 1410.3062 (24 pp.).

[19] S. R. Foguel, *The Ergodic Theory of Markov Processes*, Van-Nostrand Reinhold, New York, 1969.

[20] V. Fonf, M. Lin and A. Rubinov, *On the uniform ergodic theorem in Banach spaces that do not contain duals*, Studia Math. **121** (1996), 67–85.

[21] V. Fonf, M. Lin and P. Wojtaszczyk, *Poisson's equation and characterizations of reflexivity of Banach spaces*, Colloq. Math. **124** (2011), 225–235.

[22] M. Gordin, *Martingale – coboundary representation for a class of random fields*, J. Math. Sci. (N.Y.) **163** (2009), 363–374.

[23] W. Gottschalk and G. Hedlund, *Topological Dynamics*, AMS Colloq. Publ. vol. 36, Amer. Math. Soc., Providence RI, 1955.

[24] M. R. Herman, *L^2 regularity of measurable solutions of a finite-difference equation of the circle*, Ergodic Th. Dyn. Sys. **24** (2004), 1277–1281.

[25] S. Horowitz, *Transition probabilities and contractions of L_∞*, Z. Wahrscheinlichkeitstheorie u, Verw. Geb. **24** (1972), 263–274.

[26] Y. Ito, *Uniform integrability and the pointwise ergodic theorem*, Proc. Amer. Math. Soc. **16** (1965), 222–227.

[27] B. Jessen, A. Zygmund, and J. Marcinkiewicz, *Note on the differentiability of multiple Fourier series*, Fund. Math. **25** (1935), 217–234.

[28] J. Komlós, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hungar. **18** (1967), 217–229.

[29] I. Kornfeld, *The additive homological equation*, Functional Anal. Appl. **10** (1976), 143–144.

[30] I. Kornfeld, *Coboundaries for commuting transformations*, Illinois J. Math. **43** (1999), 528–539.

[31] I. Kornfeld and M. Lin, *Coboundaries of irreducible Markov operators on $C(K)$*, Israel J. Math. **97** (1997), 189–202.
[32] G. Kozma and N. Lev, *Exponential Riesz bases, discrepancy of irrational rotations and BMO*, J. Fourier Anal. Appl. **17** (2011), 879–898.
[33] U. Krengel, *Ergodic Theorems*, De Gruyter, Berlin, 1985.
[34] V. P. Leonov, *On the dispersion of time-dependent means of a stationary stochastic process*, Theory of Prob. and its Appl. **6** (1961), 87–93.
[35] M. Lin, *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. **43** (1974), 337–340.
[36] M. Lin, *On quasi-compact Markov operators*, Ann. Probab. **2** (1974), 464–475.
[37] M. Lin and R. Sine, *Ergodic theory and the functional equation $(I-T)x = y$*, J. Operator Theory **10** (1983), 153–166.
[38] J. Moulin Ollagnier and D. Pinchon, *A note about Hedlund's theorem*, Astérisque **50** (1977), 311–313.
[39] W. Parry and K. Schmidt, *A note on cocycles of unitary representations*, Proc. Amer. Math. Soc. **55** (1976), 185–190.
[40] A. Quas, *Rigidity of continuous coboundaries*, Bull. London Math. Soc. **29** (1997), 595–600.
[41] E. A. Robinson, *Sums of stationary random variables*, Proc. Amer. Math. Soc. **11** (1960), 77–79.
[42] R. Sato, *A remark on real coboundary cocycles in L^∞-spaces*, Proc. Amer. Math. Soc. **131** (2003), 231–233.
[43] R. Sato, *On solvability of the cohomology equation in function spaces*, Studia Math. **156** (2003), 277–293.
[44] D. Volný, *BV coboundaries over irrational rotations*, Studia Math. **126** (1997), 253–271.
[45] K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.
[46] A. Zygmund, *Trigonometric series*, Corrected 2nd ed., Cambridge University Press, Cambridge (UK), 1969.

Department of Electrical and Computer Engineering, Ben-Gurion University, Beer Sheva
E-mail address: guycohen@bgu.ac.il

department of Mathematics, Ben-Gurion University, Beer Sheva
E-mail address: lin@math.bgu.ac.il