**Homework Set #5**
**Polar codes**

1. **Polarization and the idea of polar codes**: The question is about polarization effect in memoryless channels that can lead to simple coding schemes that achieve the capacity which are called polar codes.

   (a) Consider the channel in Fig. 1 where two parallel binary erasure channels can be used at once (the input is $X = (X_1, X_2)$). The inputs alphabets are binary, so that $Y_1$ and

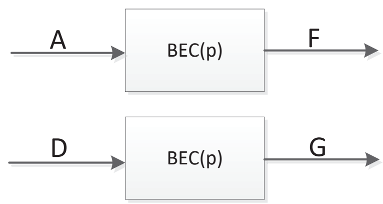


Figure 1: Two parallel binary erasure channels

   $Y_2$ are the outputs of a BEC($p$) with inputs $X_1$ and $X_2$, respectively. Compute the capacity of this channel, namely,

$$\max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2). \tag{1}$$

   What is the input distribution $p(x_1, x_2)$ that achieves the capacity?

   (b) Consider the system in Fig. 2, where addition is modulo 2:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

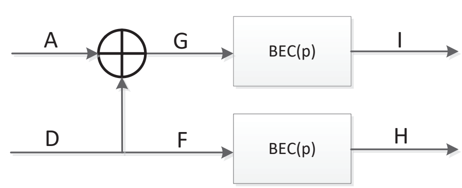


Figure 2: Two parallel binary erasure channels with modified inputs

Compute the capacity of the new channel, i.e. $\max_{p(u,v)} I(U,V;Y_1,Y_2)$.

What is the $p(u,v)$ that achieves the capacity?

**Next, the channel is decomposed into two parallel channels as appears in Fig. 3. The input of Channel** $1$ **is** $U$ **and its output is** $(Y_1,Y_2,V)$**. The input of Channel** $2$ **is** $V$ **and its output is** $(Y_1,Y_2)$**.**

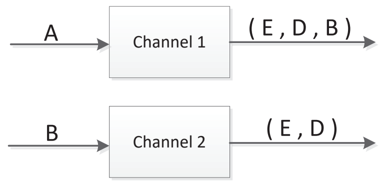


Figure 3: Two new channels

(c) Compute the expressions $I(U;Y_1,Y_2,V)$ and $I(V;Y_1,Y_2)$ with respect to the $p(u,v)$ that achieves the maximum in (b). What is the sum of the expressions you computed?

(d) Compare the mutual information of Channels 1 and 2 with the capacity of a binary erasure channel (that is, write $<$, $>$ or $=$ with simple proof).

*For large $n$, repeating this decomposition $n$ times, ends up in $nc$ clean channels and in $n(1-c)$ totally noisy channels. This is the main idea of polar codes, which achieves capacity.

2. **Polar codes** Consider a binary erasure channel $W$ with erasure probability $p$. One step of the polarization process creates a better channel $W^+$ and a worse channel $W^-$ from two independent copies of $W$.

(a) The polar code creates 4 effective channels $W^{++}$, $W^{+-}$, $W^{-+}$, $W^{--}$. Write down the capacities of these 4 channels in terms of information-theoretic quantities.

(b) Compute explicitly the capacities of the 4 channels in terms of the parameter $p$.

(c) Suppose we would like to send at the rate $3/4$ bits per channel use. Which of the $U_i$'s should be frozen, and which should be set as information bits?

(d) We repeat the polarization process $n$ times to create $2^n$ different channels from $2^n$ compies of W. Let $\overline{W}$ and $\underline{W}$ be the best and worst channels among these $2^n$ channels. Compute explicitly the capacities of $\overline{W}$ and $\underline{W}$ in terms of the parameters $p$ and $n$.

(e) What happens to the capacities of $\overline{W}$ and $\underline{W}$ as $n \to \infty$?

3. **Polar compressor** For a positive integer $N$, let $n = 2^N$ and consider the invertible matrix $P_n \in \mathbb{F}_2^{n \times n}$ defined by:

$$P_n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes N}.$$

Further, consider $Z^n = (Z_1, \ldots, Z_n) \sim \text{Bern}(p)^n$ where $p \in (0, 0.5)$, and let $W^n = Z^n \cdot P_n$.

(a) For both channel coding and source coding, we do polarization. Explain briefly the difference in polarization between channel coding and source coding.

(b) Define the rate for source coding and channel coding, and explain whether you want to maximize or minimize those rates.

(c) Assume $n = 4$ and consider the entropy terms $H(W_i|W^{i-1})$ for $i \in \{1, \ldots, 4\}$. Determine and explain which one is the highest, and calculate this specific entropy term explicitly in terms of $p$.

(d) Define the set $S_\tau$ as follows:

$$S_\tau = \{i \in \{1, \ldots, 4\} \mid H(W_i|W^{i-1}) \geq \tau\}.$$

For $\hat{\tau} = -\mathbb{E}[\log_2(P_{W_1})]$, write explicitly the set $S_{\hat{\tau}}$. Explain your result.

(e) Consider $z^4 = [1, 0, 1, 1]$ and the set $S_{\hat{\tau}}$ that you found in the previous item. What is the output of the encoder?

(f) This time let $n = 2$, and assume that $Z_1 \sim \text{Bern}(p_1)$ and $Z_2 \sim \text{Bern}(p_2)$ are sampled conditioned on $Z_1 + Z_2 = a$ (for $a \in \mathbb{F}_2$). Let $b(p_1, p_2, a)$ denote the probability of $Z_2$ being 1 conditioned on $Z_1 + Z_2 = a$. Find $b(p_1, p_2, a)$ for both $a = 0$ and $a = 1$.

)