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Application of Linear Programming to the Economical Optimization of Electrical Networks

Abstract—**The problem of specifying tolerances of electrical components used to construct a desired network to minimize network cost is translated into a linear programming format. This program enables the designer to select the tolerances according to economic factors without violating design objectives. The program employs sensitivity functions and assumes linearity over the range of operation.**

The introduction of large, high-speed digital computers has stimulated many investigators to explore new computer-compatible techniques for the analysis, design, and optimization of electrical networks [1]. These techniques, however, usually deal with models and idealized parameters and do not handle the problem of implementing the solutions into the real world.

The engineering problem of constructing an electrical network involves two phases: the design cycle and components specification. The ideal values, obtained in the first stage, must be "softened" by specifying permissible tolerances so that real components can be used. Furthermore, a major goal of the designer is to reduce the overall cost of the network by specifying broad tolerances of the components without violating, of course, the design objectives.

LINEARIZATION

Fig. 1 is a schematic presentation of the price-tolerance function of electrical components. The range of use is usually restricted since there is a practical lower limit below which the price of the component becomes prohibitive. The higher end is dictated by the technology of production since broadening of tolerances, above a certain limit, does not lower the price of the component.

We shall approximate the function of Fig. 1 by a linear function such that the price of a component i can be expressed by

$$P_i = P_{\max_i} - \xi_i \theta_i \tag{1}$$

where

P_i = price of component i (value)
$P_{\max_i}$ = maximum practical price of component i (value)
ξ_i = tolerance of component i above a minimum practical tolerance (percent)
θ_i = price reduction constant (value/percent).

Each component is associated with some minimum and maximum tolerances $T_{\min_i}$, $T_{\max_i}$ as explained above. Note that the "practical-ideal" sys-

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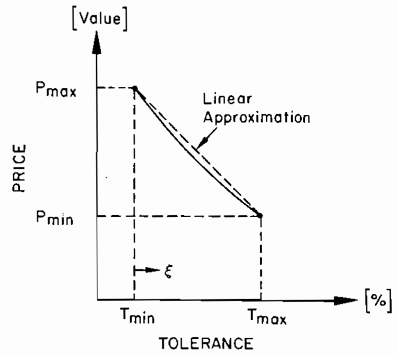

Fig. 1. Schematic presentation of price-tolerance function of electrical components.

tem will have components with tolerances

$$T_{\min_1}, T_{\min_2}, T_{\min_3}, \cdots, T_{\min_n}.$$

The amount of money saved by specifying ξ_i above $T_{\min_i}$ is clearly $\xi_i\theta_i$ and the optimality criterion is, therefore, to maximize

$$\sum_{i=1}^{n} \xi_i\theta_i \qquad i = 1, 2, \cdots, n. \tag{2}$$

The various characteristics of the network (such as gain, frequency response, etc.) are functions of the value of the components. Although these functions are nonlinear, in general, they can be linearized over a limited range using the sensitivity concept [2].

The sensitivity δ_{ij} of a characteristic j to a component i is defined by

$$\delta_{ij} = \frac{dK_j}{dC_i}\frac{C_i}{K_j} \tag{3}$$

where K_j = value of characteristic j, and C_i = value of component i.

For small changes of C_i,

$$\alpha'_{ij} \cong |\delta_{ij}|(\xi_i + T_{\min_i}) \tag{4}$$

where α'_{ij} is the maximum percentage change of characteristic j which may result from imperfection of component i, or

$$\alpha_{ij} \cong |\delta_{ij}|\xi_i \tag{5}$$

where

$$\alpha_{ij} = \alpha'_{ij} - |\delta_{ij}|T_{\min_i}. \tag{6}$$

It should be emphasized that the ξ_i are small numbers and, therefore, (4) and (5) can be used to approximate the actual situation.

An upper bound on the change in the value of characteristic j will be

$$\sum_{i=1}^{n} \alpha_{ij} \qquad i = 1, 2, \cdots, n \tag{7}$$

assuming that the tolerances ξ_i, $T_{\max_i}$, $T_{\min_i}$ are real positive numbers. This upper bound should be smaller than a given permissible tolerance β'_j, namely,

$$\sum_{i=1}^{n} \alpha'_{ij} \leqq \beta'_j \tag{8}$$

where

$$\beta'_j \geqq \sum_{i=1}^{n} T_{\min_i}|\delta_{ij}|$$

or

$$\sum_{i=1}^{n} \alpha_{ij} \leqq \beta_j \tag{9}$$

where

$$\beta_j = \beta'_j - \sum_{i=1}^{n} T_{\min_i}|\delta_{ij}|. \tag{10}$$

Therefore, the design constraints can be summarized as

$$\sum_{i=1}^{n} \alpha_{ij}\xi_i \leqq \beta_j \qquad j = 1, 2, \cdots, m. \tag{11}$$

Linear Programming Formulation

The network optimization problem can now be restated in linear programming (LP) format [3].

Find:

$$\xi_i \geqq 0 \qquad i = 1, 2, \cdots, n$$

to maximize:

$$\sum_{i=1}^{n} \xi_i\theta_i \tag{12}$$

subject to:

$$\sum_{i=1}^{n} \alpha_{ij}\,\xi_i \leqq \beta_j \qquad j = 1, 2, \cdots, m$$

and

$$\xi_i \leqq (T_{\max_i} - T_{\min_i}) \qquad i = 1, 2, \cdots, n.$$

The Dual Problem

The dual problem [3] of the LP problem will be as follows.

Find:

$$\eta_j \geqq 0 \qquad j = 1, 2, \cdots, m$$

to minimize:

$$\sum_{j=1}^{m} \beta_j\eta_j + \sum_{i=1}^{n} (T_{\max_i} - T_{\min_i})\eta_{m+i}$$

subject to:

$$\sum_{j=1}^{m} \alpha_{ij}\eta_j + \eta_{m+i} \geqq \theta_i \qquad i = 1, 2, \cdots, n. \tag{13}$$

Since θ_i has the dimension (value/percent), and α_{ij} is dimensionless, η_i should have the same dimension.

Application of the duality theorem [3] reveals that both the primal and the dual problems have optimal solutions. The primal has a feasible solution since $\xi_i = 0$ is a solution. The dual also has a feasible solution since η_i can be made large enough to satisfy the constraints of (13). Hence, from the duality theorem, both the primal and the dual have optimal solutions.

Computation

The practicability of our LP approach to the problem of economical optimization clearly depends on the question of whether or not such a scheme can be handled by a computer. In practice, electrical networks may include a large number of components, which may exclude manual computation.

The most complicated computational problem is probably the calculation of the sensitivity coefficients. It has been shown, however, that voltage and current sensitivities of a general network may be obtained by constructing a source-coupled auxiliary network [5], [6]. The computation required may be carried out with many general-purpose digital network-analysis programs without modification of the program. For sensitivity coefficients other than voltage or current sensitivities, a straightforward approach may be adopted, namely, perturbating the components around their nominal values and computing the resulting change in the characteristic. This scheme of computation may also be carried out by operational computer network-analysis techniques [1].

The simplex method [3] is an efficient technique for solving LP problems. A number of digital computer programs have already been prepared to solve LP programs by the simplex or improved simplex algorithm [4]. Most of the programs can handle problems with approximately 100 variables and 100 equations. A few of the programs, however, will handle any number of variables and equations. It seems, therefore, that the computation of the LP problem imposes no special problem for digital computations.

This approach can thus be readily implemented by slightly modifying existing computer programs. Furthermore, it might be beneficial to add the optimization program to existing network-analysis and design computer programs in order to enable the designer to specify component tolerances according to an economic key.

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