GREEN FUNCTIONS, COVARIANCE OPERATORS AND CANONICAL STOCHASTIC FIELDS

Bas Michielsen
ONERA, Toulouse, France

Cécile Fiachetti
ENSEEIHT, Toulouse, France

Abstract: In this paper, we establish, within the context of classical, frequency domain, electromagnetics, a general relation between the Green functions of a lossless configuration and the spatial covariance of a specific ("canonical") stochastic field in that configuration. As it appears, the canonical stochastic field of free space, according to our definitions, is the well-known isotropic plane wave. Application of the ideas to the case of a semi-infinite short-circuited waveguide, leads to a new canonical stochastic field which is important for the analysis of reverberating environments like Mode Stirred Chambers.

INTRODUCTION

It has been shown that the spatial covariance of an isotropic stochastic plane wave is in fact the real part of the Green tensor function of free space (see [1, 2]). The proof of this relation is based on a straightforward elaboration of the definition of covariance to the stochastic field considered. The question arises naturally, whether such a relation can be generalised to other geometrical configurations and stochastic fields satisfying special boundary conditions. In this paper, we establish a precise relation between the two objects and derive the necessary conditions such that these relations do exist indeed. The universality of the relations makes us call stochastic fields, satisfying this relation in a given configuration, the “canonical” stochastic field in the configuration. A specific application to a semi-infinite short-circuited waveguide is elaborated, showing that the definition of a canonical stochastic field leads to a pseudo-isotropic plane wave useful for the modelling of mode stirred chamber interaction problems.

FUNDAMENTAL INTEGRAL RELATIONS

Let \{E, H, J\} \textsuperscript{a,b} be two electromagnetic states in a given configuration with reciprocal and lossless media. Let the support of the two current distributions be contained in some bounded subdomain, D, of the configuration, then we have the Lorentz field reciprocity relation

\[ \int_D n \cdot (E^a \times H^b - E^b \times H^a) = - \int_D (E^a \cdot J^b - E^b \cdot J^a) \]  

(1)

and the power reciprocity relation,

\[ \int_D n \cdot (E^a \times \overline{H^b} + \overline{E^b} \times H^a) = - \int_D (E^a \cdot \overline{J^b} + \overline{E^b} \cdot J^a) \]  

(2)

Using the fact that the two states satisfy the same, source-free, Maxwell equations and boundary/radiation conditions outside D, one shows that the surface integral on the left hand side of (1) vanishes. Let us label the electric field in \( \Omega \) by the current distribution which generates it. The field reciprocity, then, learns that

\[ \int_{\partial D} (E^a \cdot J^b - E^b \cdot J^a) = 0 \]

If we subtract this relation from the power reciprocity relation, we get,

\[ \int_{\partial D} n \cdot (E^a \times \overline{H^b} + \overline{E^b} \times H^a) = - \int_D (E^a \cdot \overline{J^b} + \overline{E^b} \cdot J^a) = (J^a, C J^b)_{L^2(D)} \]  

(3)

Where we used \( E_J(x) = \int_{y \in D} G_e^{x,y}(y) J(y) \), with \( G^{x,y} \) a Green tensor function for the configuration, and \( C \) is the real, symmetric operator with kernel distribution \(-2 \text{Re}[G^e]\) and \( (.,.)_{L^2(D)} \) the standard inner product in \( L^2(D) \).

NATURAL FIELD DECOMPOSITIONS AND SCATTERING THEORY

We shall introduce a decomposition of space \( \mathbb{R}^3 = \Omega^- \cup \Omega^+ \) where \( \Omega^- \) is a bounded interior and \( \Omega^+ \) is an unbounded exterior. For ease of presentation, we consider \( \varepsilon = \varepsilon_0 \) and \( \mu = \mu_0 \) throughout \( \Omega^+ \) and in an open

URSI EMTS 2004 299
neighbourhood of $\partial\Omega^-$. Any solution of the Maxwell equations in $\Omega = \Omega^- \cup \Omega^+$, can be uniquely decomposed into two constituents

$$\{ E, H \} = \{ E^-, H^- \} + \{ E^+, H^+ \}$$

where $\{ E^\pm, H^\pm \}$ satisfy the source-free Maxwell equations in $\Omega^\pm$, respectively. The boundary limits on $\partial\Omega^-$ of these fields are in the function space $X^\pm \ni (E^\pm_n, n \times h^\pm_p)$ ($n$ is the outward pointing normal on $\partial\Omega^-$). It will be shown that there exist, in general, two related systems of basis fields, $\{ e^\pm_n, n \times h^\pm_p \} \subset X^\pm$, parameterised by their traces on $\partial\Omega^-$, suitably orthogonalised and such that

$$\int_{\partial\Omega^-} n \cdot (e^+_p \times H^+ + E^+ \times h^+_p) = \int_{\partial\Omega^-} n \cdot (e^-_p \times H^- - E^- \times h^-_p)$$

(4)

DEFINITION OF A CANONICAL STOCHASTIC FIELD

Let $J$ be an oscillating current distribution with bounded support in $\Omega^-$, generating a time harmonic electromagnetic field, $\{ E, H \}$. This field has an $X^+$ expansion in $\Omega^+$, the amplitudes of which are given by,

$$A^+_p = \int_{\partial\Omega^-} n \cdot (e^+_p \times H^+ + E^+ \times h^+_p) = \int_{\partial\Omega^-} -e^-_p \cdot J$$

(5)

where we used eq.’s (4) and (1). Substituting the outgoing wave expansion into (3) we find

$$\langle J^a, CJ^b \rangle_{L^2(\Omega)} = \sum_{pq} A^+_{ap} \overline{A^-_{aq}} \int_{\partial\Omega^-} n \cdot (e^+_p \times h^-_q + e^-_p \times h^+_q) = \sum_{p} A^+_{ap} \overline{A^-_{ap}}$$

(6)

This puts in direct correspondence the inner product of the outgoing wave expansion and the special inner product between the sources of the radiation field. We now define a “canonical” stochastic electromagnetic field, $\{ E^-, H^- \} = \sum_p A^-_p \{ e^-_p, h^-_p \}$ where $\{ e^-_p, h^-_p \}$ is an element of the basis in $X^-$ introduced in section 3 and the $A^-_p$ are independent and identically distributed (IID) stochastic amplitudes with variance $\sigma^2$. We can study the properties of the covariance operator of the electric field, $C_E^-$, i.e., the operator with kernel $E(E^-(x)E^-(y))$, by evaluating it on two arbitrary current distributions $J^a$ and $J^b$, like $\langle J^a, C_E^-, J^b \rangle$. Substitution of the wave expansion gives,

$$\langle J^a, C_E^-, J^b \rangle = \sum_{pq} \overline{E(A^-_p \overline{A^-_q})} \langle e^-_p, J^a \rangle \langle e^+_p, J^b \rangle = \sum_{p} \sigma^2 \langle e^-_p, J^a \rangle \overline{\langle e^-_p, J^b \rangle} = \sigma^2 \sum_{p} A^+_{ap} \overline{A^-_{ap}}$$

(7)

where we used $E(A^-_p \overline{A^-_q}) = \sigma^2 \delta_{p-q}$ and eq. (5). Comparison of this last equation with equation (6), which both hold for arbitrary current distributions, shows that we can identify the operator $C$, introduced above with the spatial correlation operator of the specific stochastic incident field we defined above.

$$C_E^- = \sigma^2 C$$

(8)

The universality of the operator $C$ justifies the qualifier “canonical” for the stochastic field thus defined.

CANONICAL STOCHASTIC FIELD IN A SHORT-CIRCUITED SEMI-INFINITE WAVEGUIDE

In this section, we elaborate an example which shows how the analysis of the previous sections can be adapted to configurations which are not truly three dimensional. The semi infinite waveguide is represented by a cylindrical domain, $\Omega = A \times \mathbb{R}_+$, with constant cross-section $A \subset \mathbb{R}^2$. A point in $x \in \Omega$ will be written as $(x_T, z)$ with $x_T \in A$ and $z \in \mathbb{R}_+$. The cross-sectional domain at axial coordinate $z$ will be referred to as $A_z$. We assume the waveguide walls, $\partial A \times \mathbb{R}_+$ and the short-circuit plane, $A_0$, to be perfectly conducting and the constitutive parameters are those of vacuum throughout $\Omega$. We choose $\Omega^- = A \times (0, L)$ and $\Omega^+ = A \times \{ \mathbb{R}_+ + L \}$. 

300 URSI EMTS 2004
In this waveguide configuration, we have the following field representation,

\[ E(x) = \sum_p [\nabla \times A_p^h(x) + j\omega \mu_0 A_p^c(x) - \frac{1}{j\omega \varepsilon_0} \nabla \cdot A_p^c(x)] \]  

(9)

and similarly for the magnetic field. The vector potentials \( A_p^c \) for the TM modes, and \( A_p^h \) for the TE modes, are defined by (we suppress the index \( p \) here, for ease of notation) \( A^h(x_T, z, \varphi) = N \phi^h(x_T) e^{jz}\gamma e^{j\varphi} \), and \( A^c(x_T, z) = N \phi^c(x_T) e^{jz}\gamma e^{j\varphi} \), with \( \phi^h|_{\varphi = 0} = 0 \) or \( \phi^h|_{\varphi = \pi} = 0 \) respectively, \( \partial_z^2 \psi^{c,b} = -\gamma^2 \psi^{c,b} \) and

\[ \gamma = \sqrt{k^2 - k_0^2}, \] with \( k_0 = \omega/\varepsilon_0 \), is called the propagation coefficient. The phase factors \( \psi^{c,b} \) determine the axial dependency of the mode fields. For the \( X^+ \)-field expansion we choose \( \psi^c(z) = \psi^h(z) = e^{-\gamma z} \). For the \( X^- \)-field expansion, we choose \( \psi^c(z) = \cos(-\gamma z) \) and \( \psi^h(z) = \sin(-\gamma z) \) such that the tangential electric field vanishes on \( A_0 \). The factor \( N \) is a normalisation coefficient, which we have to adjust in order that the desired orthogonalisation in section 3 is obtained. In the lossless waveguide configuration, we have modes which are either propagating modes, labeled by the index set \( P \), or evanescent modes, indexed by the set \( E \).

The existence of evanescent modes marks an important difference with the truly three dimensional case, i.e., non-trivial fields exist in \( \Omega^+ \) which do vanish rapidly at infinity. As a consequence, we have

\[ \forall q \in E \left[ \int_{A_0} i_z \cdot (e^+_q \times h^+_q + e^-_q \times h^-_q) = 0 \right] \quad \text{and} \quad (J^p_i, C_i q)_L(A) = \int_{A_0} i_z \cdot (E^p \times H^+ + E^p \times H^-) = \sum_{p \in P} A_p^{+\sigma} A_p^{-\delta} \]

Therefore, if we maintain our definition of a canonical stochastic field from the general theory, we should construct such a field as the finite superposition, with statistically independent and identically distributed coefficients, of the appropriately normalised \( X^- \)-fields spanned by the propagating modes only.

It will be shown that the following normalisation factors yield basis fields satisfying the constraints,

\[ N^{TE,+} = \frac{\sqrt{\varepsilon_0 \mu_0}}{2 \beta k_c \|\phi^h\|^2_{L^2(A)}} = \frac{1}{2} N^{TE,-} \quad \text{and} \quad N^{TM,+} = \frac{\sqrt{\varepsilon_0 \mu_0}}{2 \beta k_c \|\phi^c\|^2_{L^2(A)}} = \frac{1}{2} N^{TM,-} \]

where \( \beta = \text{Im}(\gamma) \). The canonical incident field in \( \Omega^- \) is the linear combination of these \( X^- \) basis fields with IID stochastic coefficients. For a rectangular cross-section \( A = (0,a) \times (0,b) \), any propagating mode is a superposition of 8 plane waves. The electric field of such a plane wave, in the principal octant, is given, in spherical coordinates, by \( E = (jN^{TE,-} A_{mn}^{TE} / 8i\phi + N_{mn}^{TM,-} A_{mn}^{TM} / 8i\phi) \) and the variances by

\[ \text{var}(E_\varphi) = \sigma^2 \frac{Z_0}{32 \|\phi^h\|^2_{L^2(A)} \cos(\theta)} \quad \text{and} \quad \text{var}(E_\theta) = \sigma^2 \frac{Z_0}{32 \|\phi^c\|^2_{L^2(A)} \cos(\theta)} \]

where \( \sigma^2 = \text{var}(A_{mn}^{TE}) = \text{var}(A_{mn}^{TM}) \), by definition of the canonical field, and \( \cos(\theta) = \sqrt{1 - k_i^2/k_0^2} \) is the cosine of the angle between the plane wave propagation direction and the positive waveguide axis.

It will be shown that these variances are such that the lack of possible propagation directions in solid angles close to the equator is “compensated” for by an increased variance. We might, therefore, call this a pseudo-isotropic stochastic plane wave.

References
