1. Introduction.

For non-stationary problems related with propagation of wave packets, or waves modulated in frequency and amplitude, the space-time ray method seems to be an adequate mathematical tool, see Lewis (1964), Babich, Buldyrev and Molotkov (1985). However, this method fails on caustics where ray amplitudes become singular just as it happens to ray method in frequency domain. It is known now that caustic problems can be overcome with the help of Gaussian beams.

The method of summation of Gaussian beams in frequency domain, suggested by Popov (1982), accumulates actually years long investigations on open resonators and approximate solutions of various type of wave equations concentrated in the vicinity of 1D cycles which have been carried out in St.Petersburg at Steklov Mathematical Institute and State University in the late 1960’s and early 1970’s, see review paper by Babich and Popov (1989). This method has proved to be efficient for stationary problems and can be extended to non-stationary ones by means of Fourier transform on frequency, however, implementation of Fourier transform leads in general to time-consuming calculations.

In this contribution we would like to draw attention to an alternative, suggested by Popov (1990) method which enables one to overcome caustic problems in rather general cases of inhomogeneous media with smooth interfaces. The method is based on summation of specific space-time Gaussian beams, we call them quasi-jets, which provide an asymptotics of the wave field in the time domain and does not involve Fourier transform. This method can be used also for problems with moving sources and interfaces. Below we describe main ideas of this method on the following point source problem for the scalar wave equation

\[ \Delta P - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} P = f(t) \delta(M - M_o) \]  

where \( \Delta \) is the Laplace operator and the wavelet \( f(t) \) describes a time-pulse modulated in amplitude and frequency

\[ f(t) = \text{Re} \{a(t) \exp(ip\Theta(t))\}. \]

Here \( a(t) \) is a smooth envelope of the pulse, \( p \) is a large parameter, i.e. \( p \to \infty \), and the derivative \( \Theta'(t) \) of the phase function does not vanish on the time duration of the pulse. We assume that \( \Theta'(t) \leq \text{const} < 0 \). In this case the instant frequency of the time-pulse \( \omega(t) = -p\Theta'(t) \) is the large parameter of the problem. In the sequel, without loss of generality, we assume that \( f(t) = 0 \) for \( t < 0 \) and \( P(M,t) = 0 \) for \( t < 0 \) as well.

2. Construction of a space-time Gaussian beam, quasi-jet.

We start with the space-time anzats for the homogeneous wave equation (1) and consider further only the leading term. In this case a solution of the wave equation we seek in the form

\[ U(\vec{x},t) \equiv U_o(\vec{x},t) \exp(ip\Theta(\vec{x},t)) \]
where \( \vec{x} = (x_1, x_2, x_3) \) and \( x_1, x_2 \) and \( x_3 \) are the Cartesian coordinates. By inserting expression (3) into the wave equation, we obtain the eikonal and transport equations for phase function \( \Theta(\vec{x}, t) \) and amplitude \( U_0(\vec{x}, t) \), respectively,

\[
(\nabla \Theta)^2 - \frac{1}{C^2} \left(\frac{\partial \Theta}{\partial t}\right)^2 = 0, \quad 2(\nabla \Theta, \nabla U_0) - \frac{2}{C^2} \frac{\partial \Theta}{\partial t} \frac{\partial U_0}{\partial t} + U_0(\Delta \Theta - \frac{1}{C^2} \frac{\partial^2 \Theta}{\partial t^2}) = 0,
\]

(4)

We would like to remind that characteristics of the eikonal equation are called the bicharacteristics of the wave equation or the space-time rays. Projection of the bicharacteristics onto space are called the rays.

Let us fix a space-time ray and assume it to be given in the form \( t = \tau(s), \vec{r} = \vec{r}(s) \) where \( \vec{r} \) is the radius-vector in 3D, \( s \) is arc length along the ray and \( \tau(s) \) is the eikonal along the ray. We shall regard it as a central space-time ray of a Gaussian beam or quasi-jet which we want to construct. With this aim, we introduce the ray-centered coordinates \( s, q_1, q_2 \)

\[
\vec{r}_M = \vec{r}(s) + q_1 \vec{e}_1(s) + q_2 \vec{e}_2(s)
\]

(5)

where \( \vec{r}_M \) is the radius-vector of an arbitrary point \( M \) in the vicinity of ray, \( \vec{e}_1(s) \) and \( \vec{e}_2(s) \) are mutually orthogonal unit vectors which are orthogonal to the ray \( \vec{r}(s) \) at any point \( s \). Thus, in the vicinity of the space-time ray we obtain the curvilinear local coordinates \( t, s, q_1, q_2 \). On the next step we use them for constructing an approximate solution of the eikonal and transport equations (4) in the form of power series on \( q_1, q_2 \)

\[
\Theta(t, s, q_1, q_2) = \Theta_0(t, s) + \frac{1}{2} (\Theta_2(t, s) q_1, q_2) + \ldots
\]

(6)

\[
U_0(t, s, q_1, q_2) = U_0^{(0)}(t, s) + \ldots
\]

where vector \( q = (q_1, q_2) \) is formed by the coordinates \( q_1, q_2 \), and \( \Theta_2(t, s) \) is a symmetric 2x2 matrix depending upon coordinates \( t, s \). Note that the terms written down in (6) correspond to the leading term of a space-time Gaussian beam we are looking for. On substituting expansions (6) into (4) we obtain, after some mathematics, the following expression for the space-time Gaussian beam

\[
U(t, s, q_1, q_2) = \frac{C_0(s)}{\sqrt{\det Q}} \exp\left\{ip[\Theta_0(t - \tau(s)) - \frac{1}{2} \Theta_0'((t - \tau(s)) (PQ^{-1} \vec{q}, \vec{q}))]\right\},
\]

(7)

where \( C_0(s) \) is the velocity calculated on the central ray \( \vec{r}(s) \). Note that \( \Theta_0(t - \tau(s)) \) remains an arbitrary smooth function. Two complex valued matrices \( P(s) \) and \( Q(s) \) are solutions of the equations in variations which are the same as in the case of stationary Gaussian beams

\[
\frac{d}{ds} Q = C_0(s) P, \quad \frac{d}{ds} P = -C_0^{-2}(s) Q,
\]

(8)

where symmetric 2x2 matrix \( C_{i,k}(s), i, k = 1, 2 \) is formed by second order derivatives of the velocity on \( q_1, q_2 \) calculated on the central ray \( q_1 = q_2 = 0 \). It can be proved that the Gaussian beam (7) has no singularities on caustics due to \( \det Q(s) \neq 0 \) for all \( s \) just as it holds in the stationary case. But now Gaussian beam (7) is an asymptotic solution of the wave equation in the time domain. Unlike introduced by Babich and Ulin (1984) quasi-photons which are
concentrated in all three space coordinates, the quasi-jets are concentrated with respect to only two space coordinates \( q_1, q_2 \).

3. Summation of space-time Gaussian beams.

Return now to the inhomogeneous wave equation (1). The point source gives rise to the family of rays emanated from the point under all possible angles. These rays can be parameterized by the spherical angles \( \nu \) and \( \phi \). Denote them by \( \vec{r} = \vec{r}(s; \nu, \phi) \) and introduce the eikonal \( \tau(s; \nu, \phi) = \int C^{-1}(s)ds \) for them. Then, the corresponding family of space-time rays takes the form

\[
t = \tau(s; \nu, \phi) + t_0, \quad \vec{r} = \vec{r}(s; \nu, \phi),
\]

where additional third parameter \( t_0 \) is the emanating time of the rays from the source. To perform summation of Gaussian beams we proceed as follows. For fixed \( t_0 \) we build the family of space-time rays and construct a Gaussian beam propagating along each ray. Then we integrate Gaussian beams on the ray parameters \( \nu \) and \( \phi \) with an initial amplitude \( \Phi(t_0, \nu, \phi) \). It results in a set of integrals depending upon parameter \( t_0 \). By excluding this parameter with the help of equation (9), we obtain the asymptotics of the initial problem in terms of space-time Gaussian beams. Just as in stationary case, the initial amplitudes come out from matching of the Gaussian beam integral with ray asymptotics of the wave field in the vicinity of source where corresponding ray field is regular. In the case under consideration, the final integral takes the form

\[
P(M, t; M_0) = \text{Re}\left\{ \frac{i\hbar}{8\pi^2} \times \right. \]

\[
\int \int \left[ \frac{C_0(s)\det Q(0)}{C(M_0)\det Q} \left| \Theta(t - \tau) a(t - \tau) \exp\left[ i\hbar \left( \Theta(t - \tau) - \frac{1}{2} \Theta'(t - \tau)(\mathbf{PQ}^{-1}\vec{q}, \vec{q}) \right) \right] \right| ds \right. \]

where the integration is over unit sphere. Note that functions \( \Theta \) and \( a \) are defined by the wavelet (2).

Recent applications of this method will be presented at the Symposium, for geophysical applications see Kachalov and Popov (1988).

References.


