Probabilistic Approach to Wave Propagation

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Abstract: The probabilistic approach to wave propagation starts similarly to the ray theory, from the representation of the wave field as a product of the amplitude and of the exponent of the eikonal which is computed by a canonical technique of analytical mechanics. However, the amplitude is not approximated but is represented by exact probabilistic formulas that admit efficient numerical evaluation and appear as a direct improvement of many asymptotic solutions. This approach is shown to be an effective tool for the analysis of numerous problems including that of wave diffraction by a screen occupying a plane angular sector.

In the theory of wave propagation, solutions of the Helmholtz equation
\[ \nabla^2 \Phi(x) + k^2 \kappa^2(x) \Phi(x) = \Upsilon(x), \quad k = \text{const}, \] (1)
are customarily sought in the product form
\[ \phi(x) = \phi(x) e^{i k S(x)}, \] (2)
where the phase \( S(x) \) satisfies the eikonal equation
\[ [\nabla S(x)]^2 = \kappa^2(x), \] (3)
from which it also follows that the amplitude \( \phi(x) \) must satisfy the complete transport equation
\[ \frac{\sigma^2}{2k} \nabla^2 \phi + \vec{A} \cdot \nabla \phi + B\phi + F = 0, \] (4)
where
\[ \sigma = \sqrt{i}, \quad \vec{A} = -\nabla S, \quad B = -\frac{1}{2} \nabla^2 S, \quad F = -i \Upsilon e^{-ik S}/k. \] (5)

There are at least three reasons justifying the use of such representation of the wave fields: a) the eikonal equation admits constructive solution by the canonical Hamilton-Jacobi method of analytical mechanics; b) the structure of the eikonal is closely connected with the intuitively clear idea of propagation along rays; and c) in many cases the amplitude \( \phi(x) \) can be well approximated by the solution \( \phi_0 \) of the equation obtained from (4) by dropping its first term.

If the first term in (4) is dropped then the resulting first-order equation has the solution
\[ \phi_0(x) = \int_0^\infty F(\xi_t)e^{-\frac{1}{2} \int_0^t \nabla^2 S(\xi) ds} dt \equiv \int_0^\infty \sqrt{\frac{J(\xi_t)}{J(x)}} F(\xi_t) dt, \] (6)
where \( J(x) \) is a characteristic of the vector field \( \vec{S}(S) \) widely known in the literature as its ‘geometrical divergence’, and \( \xi_t \) is the solution of the ordinary differential equation
\[ d\xi_t = -\nabla S(\xi_t) dt, \quad \xi_0 = x, \] (7)
so that the trajectory of \( \xi_t \) is the ray along which the wave arrives at \( x \).

It is clear that the approximation \( \phi \approx \phi_0 \) is accurate only when \( k \gg 1 \) and when the geometrical divergence \( J(x) \) has no singularities in a vicinity of the ray passing through the observation point, which is not the case in many important situations arising, for example, in problems of propagation of low-frequency waves, problems of diffraction, and problems of wave propagation in nonhomogeneous media.
Such limitations naturally generate numerous attempts to improve the elementary approximation \( \phi \approx \phi_0 \) either by building a series \( \phi \approx \phi_0 + \phi_1 + \ldots \), or by a more complicated choice of the initial approximation \( \phi_0 \), or by a combination of both of these ideas.

Although a lot of progress has been achieved in finding asymptotic or approximate solutions of the complete transport equation (4), it is however instructive and useful to observe that the exact solution \( \phi(x) \) can be represented by explicit formulas

\[
\phi(x) = \mathbf{E} \int_0^\infty F(\xi_t) e^{\frac{R^t}{2} B(\xi_t) dt} = \mathbf{E} \int_0^\infty F(\xi_t) e^{\frac{R^t}{2} \nabla^2 S(\xi_t) ds} dt,
\]

where \( \mathbf{E} \) denotes the mathematical expectation computed over the trajectories of the random motion \( \xi_t \) governed by the stochastic differential equation

\[
d\xi_t = \frac{\sigma(\xi_t)}{\sqrt{k}} dw_t + \bar{A}(\xi_t) dt = \sqrt{\frac{1}{k}} dw_t - \nabla S(\xi_t) dt, \quad \xi_0 = x,
\]

driven by the standard Brownian motion \( w_t \). The meaning of equations like (9) is rigourously described in the theory of stochastic process [6, 7], but in many cases it suffices to view \( \xi_t \) as the limit at \( \epsilon \to 0 \) of the discrete series of random jumps

\[
\xi_t \rightarrow \xi_{t+\Delta t} \equiv \xi_t + \bar{A}(\xi_t) \Delta t \pm \frac{\epsilon \sigma(\xi_t)}{\sqrt{k}} \epsilon_j,
\]

along one of \( 2N \) equally possible Cartesian directions \( \pm \epsilon_j \). It is important to mention that the time increment \( \Delta t \) and the spatial increment \( \epsilon \) of these jumps are related to each other by \( \Delta t = \frac{\epsilon^2}{N} \).

Solutions of the type (8) are widely known as the Feynman-Kac formulas, and they are usually derived with the assumption that all of the coefficients of the equation (4) are real. However, as shown in our previous papers [1, 3, 4], the analytic solution of the equation (4) with complex coefficients from (5) can still be represented by (8).

The special relationship \( B = \text{div}(\bar{A}) \) between the coefficients \( \bar{A}(x) \) and \( B(x) \) from (5) makes it possible to convert (8) to the form

\[
\phi(x) = \mathbf{E} \int_0^\infty F(\xi_t) \sqrt{\frac{J(\xi_t)}{J(x)}} \exp \left( \int_0^t \frac{1}{4k} \nabla^2 \ln J(\xi_s) ds - \sqrt{\frac{1}{4k}} \nabla \ln J(\xi_s) dw_s \right) dt,
\]

where \( J(x) \) is the same geometric divergence that appears in (6). This expression clearly demonstrates that the exact solution of the Helmholtz equation obtained by random walks directly improves the asymptote (6). Indeed, at the high-frequency limit \( k \to \infty \) the random motion \( \xi_t \) defined by (9) degenerates to the deterministic motion (7), the exponent in (11) degenerates to unity, and (11) coincides with (6).

Probabilistic formulas like (8) and (11) admit many modifications, analogs and generalizations which make it possible to solve various problems for partial differential equations. Thus, the solution of the Dirichlet problem

\[
\sum_{n=1}^N \left( \frac{\sigma_n^2}{2} \frac{\partial^2 \phi}{\partial x_n^2} + A_n \frac{\partial \phi}{\partial x_n} \right) + B \phi = 0, \quad \phi|_{\partial G} = f, \quad \phi \in B(\xi_t) dt,
\]

formulated in \( N \)-dimensional domain \( G \) can be represented as the mathematical expectation

\[
\phi(x) = \mathbf{E} \left\{ f(\xi_t) e^{\frac{R^t}{2} B(\xi_t) dt} \right\},
\]
where $\xi_t = (\xi_t^1, \xi_t^2, \ldots, \xi_t^N)$ is the $N$-component random motion controlled by the equations

$$d\xi_t^1 = \sigma_n(\xi_t) dw_t^1 + A_n(\xi_t) dt, \quad \xi_0 = x,$$  \hspace{1cm} (14)

and stopped at the exit time $\tau$ defined as the first time when $\xi_t$ touches the boundary $\partial G$.

To illustrate the usefulness of the solution (13) consider the two dimensional problem

$$\nabla^2 \Phi(r, \theta) + k^2 \Phi(r, \theta) = 0, \quad \Phi(r, \alpha_n) = f(r, \alpha_n)e^{ikr}$$  \hspace{1cm} (15)

formulated in a wedge $G = \{r, \theta : r > 0, \alpha_1 < \theta < \alpha_2\}$. Seeking the solution in the form $\Phi(r, \theta) = \phi(r, \theta)e^{ikr}$. In this case the eikonal is defined as $S = r$, and the complete transport equation (4) gets, after multiplication by $q = -ikr^2$, the form

$$\frac{r^2}{2} \frac{\partial^2 \phi}{\partial r^2} + r \left( \frac{1}{2} + ikr \right) \frac{\partial \phi}{\partial r} + \frac{1}{2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{ikr}{2} \phi = 0,$$  \hspace{1cm} (16)

which matches the structure of (12). Consequently, the solution $\phi(r, \theta)$ is represented as the mathematical expectation

$$\phi(r, \theta) = E \{ f(\xi_t^1, \xi_t^2) \exp \left( \frac{ik}{2} \int_0^r \xi_t^1 ds \right) \},$$  \hspace{1cm} (17)

where $\xi_t^1$ and $\xi_t^2$ are independent random motions governed by the equations

$$d\xi_t^1 = \xi_t^1 dw_t^1 + \xi_t^1 \left( \frac{1}{2} + i k \xi_t^1 \right) dt, \quad \xi_0^1 = r,$$  \hspace{1cm} (18)

$$d\xi_t^2 = dw_t^2, \quad \xi_0^2 = \theta,$$  \hspace{1cm} (19)

Equation (19) shows that the angular component $\xi_t^2$ performs the standard Brownian motion on the real line, and that the radial component $\xi_t^1$ runs across the complex plane remaining in the quadrant $\text{Re}(\xi_t^1) > 0, \text{Im}(\xi_t^1) > 0$, which guarantees the convergence of (17).

Although (17) already represents the solution of (15) it is instructive to observe that the combination of (18) with Ito’s formula [6,7] $d \log(\xi_t^1) = d\xi_t^1/\xi_t^1 - \frac{1}{2} dt$ leads to the expression $ik\xi_t^1 dt = d \log(\xi_t^1) - dw_t$ whose substitution to (17) results in the alternative solution

$$\phi(r, \theta) = \frac{1}{\sqrt{r}} E \left\{ f(\xi_t^1, \xi_t^2) \sqrt{\xi_t^1} \exp \left( -\frac{w_r}{2} \right) \right\},$$  \hspace{1cm} (20)

which emphasizes the structure of $\phi(r, \theta)$ at infinity.

Although the demonstrated technique looks very simple it is powerful enough to generate explicit solutions of non-trivial two-dimensional problems of wave propagation in arbitrary wedges [3] and in the half-space with surface breaking cracks and cavities of arbitrary shape [5]. Moreover, the three-dimensional version of this technique made it possible to solve the challenging problem of diffraction by a screen occupying a plane wedge of arbitrary angle [2].

References