A SIMPLE APPROACH TO QUASI-TEM ANALYSIS OF A PLANAR MULTI-CONDUCTOR STRUCTURE EMBEDDED IN AN ELLIPTICALLY STRATIFIED ENVIRONMENT

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Abstract The quasi-TEM modes on a planar multiconductor transmission line embedded in an elliptically stratified cross section are considered. The electro- and magnetostatic problems are solved using separation of variables in elliptical coordinates. It is shown that asymptotic solutions for the radial dependences of the terms of the series can be used under certain condition on the profiles of stratification.

INTRODUCTION AND PROBLEM FORMULATION

Consider \( N_c \) coplanar thin stripline conductors located in the region \((-a < x < a) \cap (y = 0)\). The surrounding medium is isotropic, and elliptically stratified in the sense that the parameters \( \varepsilon \) and \( \mu \) are functions of the elliptic coordinate \( u \) only, where the relation between the Cartesian \((x, y, z)\) and the elliptical coordinates \((u, v, z)\) are \( x = a \cosh u \cos v \), \( y = a \sinh u \sin v \), \( z = z \). The set of planar conductors thus belongs to a part of the coordinate surface \( u = 0 \); see figure 1.

\[ \Phi(u, v) = \sum_{n=0}^{\infty} U_n(u) \cos n v, \]  

where each \( U_n(u) \) satisfies the following differential equation and boundary condition at the shielding:

\[ \left( \frac{d^2}{du^2} + \frac{1}{\varepsilon(u)} \frac{d}{du} \frac{d}{du} - n^2 \right) U_n = 0, \]  

\[ U_n(u_s) = 0,\]  

The whole configuration is enclosed within an elliptically shaped perfectly conducting shielding, at \( u = u_s \), held at a zero reference potential. Such a configuration approximates well some practical transmission lines, like e.g. shielded parallel lines.

In order to find the \( N_c \) quasi-TEM modes, one must determine the capacitance and inductance matrices, \( C \) and \( L \), by solving Laplace’s equations for the electric and magnetic potentials.

ANALYSIS

The major step is to solve the following two-dimensional electrostatic and magnetostatic problems [1]:

\[ \nabla_T \cdot (\varepsilon \nabla_T \Phi) = 0, \quad \Phi = \text{constant on each conductor,} \]  

\[ \nabla_T \cdot \left( \frac{1}{\mu} \nabla_T A \right) = 0, \quad A = \text{constant on each conductor,} \]  

where \( \nabla_T \) is the transverse part of the del-operator, \( \Phi \) is the scalar electric potential and \( A \) is the single component of the longitudinally directed magnetic vector potential \( A = A \hat{z} \). Once the electrostatic problem has been solved, the magnetostatic problem is solved using the same scheme after the replacements \( \Phi \rightarrow A, \varepsilon \rightarrow 1/\mu \). Using separation of variables, the electric potential \( \Phi \) becomes

\[ \Phi(u, v) = \sum_{n=0}^{\infty} U_n(u) \cos n v, \]  

where each \( U_n(u) \) satisfies the following differential equation and boundary condition at the shielding:

\[ \left( \frac{d^2}{du^2} + \frac{1}{\varepsilon(u)} \frac{d}{du} \frac{d}{du} - n^2 \right) U_n = 0, \]  

\[ U_n(u_s) = 0,\]
For a permittivity profile \( \varepsilon (u) \), the radial equation (4) can be solved numerically, for each value of \( n \), by integration in the \(-u\)-direction starting from the boundary condition (5) and \( \frac{dU_n}{du}(u_n) = d_n \) where \( d_n \) is an initially arbitrary real and nonzero constant; one obtains

\[
\frac{dU_n}{du}(0) = b_n U_n(0),
\]

(6)

where \( b_n \) is a constant that depends on \( \varepsilon (u) \) but not on \( d_n \). The set \( \{b_n\}_n^{\infty} \) thus contains all information about the dielectric properties of the medium surrounding the planar conductors.

We divide the surface \( u = 0 \) into the parts \( \mathcal{C} \) and \( \hat{\mathcal{C}} \), where \( \mathcal{C} \) is the part occupied with the conductors and \( \hat{\mathcal{C}} \) is the complement. For brevity, we use the notation \( f_n(v) = \cos n \pi v \) for the basis functions, which satisfy the following orthogonality relation over the interval \( v \in [0, \pi] \), i.e. \( \mathcal{C} + \hat{\mathcal{C}} \):

\[
\langle f_m, f_n \rangle_{\mathcal{C}} + \langle f_m, f_n \rangle_{\hat{\mathcal{C}}} = \int_{\mathcal{C}} f_m f_n dv + \int_{\hat{\mathcal{C}}} f_m f_n dv = \int_0^{\pi} f_m f_n dv = \frac{\pi}{2} (\delta_{mn} + \delta_{m0}\delta_{0n}),
\]

(7)

where \( \delta_{mn} = 1 \) when \( m = n \) and zero otherwise.

Let \( B_n \) denote the correct value of \( U_n(0) \). The boundary conditions thus become (cf. (3) and (6))

\[
\phi(0^+, v) = \sum_{n=0}^{\infty} B_n f_n(v) = F(v), \quad v \in \mathcal{C}
\]

(8)

\[
[\partial_v \phi](0^+, v) = \sum_{n=0}^{\infty} b_n B_n f_n(v) = 0, \quad v \in \hat{\mathcal{C}}
\]

(9)

where the function \( F(v) \) assumes a piecewise constant value; the potential at the conductor which \( v \) maps onto. A testing procedure, exploiting the relation (7), yields after some simplifications

\[
b_m \frac{\pi}{2} (1 + \delta_{m0}) B_m + \sum_{n=0}^{\infty} (b_m - b_n) \langle f_m, f_n \rangle_{\mathcal{C}} B_n = b_m \langle f_m, F \rangle_{\mathcal{C}}, \quad m = 0, 1, 2, \ldots
\]

(10)

Limiting the values of \( m \) and \( n \) in the equation system generated by (10) to a certain number \( N_c \), we obtain a matrix equation that gives approximate values of the coefficients \( \{B_n\}_{n=0}^{N_c} \).

If the \( c \)th conductor is located between \( x_c^a = a \cos v_c^a \) and \( x_c^b = a \cos v_c^b \), the charges per unit length, \( \{\lambda_c\}_{c=1}^{N_c} \), and the line currents, \( \{I_c\}_{c=1}^{N_c} \), on the conductors become

\[
\lambda_c = -2\varepsilon(0) \sum_{n=0}^{\infty} b_n^{(\Phi)} B_n^{(\Phi)} \int_{v_c^a}^{v_c^b} \cos(nv) dv, \quad c = 1, \ldots, N_c,
\]

(11)

\[
I_c = -\frac{2}{\varepsilon(0)} \sum_{n=0}^{\infty} b_n^{(A)} B_n^{(A)} \int_{v_c^a}^{v_c^b} \cos(nv) dv, \quad c = 1, \ldots, N_c,
\]

(12)

From (11) and (12) one finds readily the capacitance matrix \( \mathbf{C} \) and the inductance matrix \( \mathbf{L} \), which contain all information for any subsequent quasi-TEM analysis of the structure [1].

**THE RADIAL EQUATION**

If the conductors have small widths and/or are closely separated, one needs many basis functions (i.e. a large \( N_c \)) to recapture the potential- and charge/current-distributions in the plane \( u = 0 \). However, since the radial equation (4) must be solved for each value of \( n \), numerical solutions can be rather time-consuming to obtain, especially for large values of \( n \) when the solutions become rapidly varying.

**Exponential stratification.** With an exponential stratification in the \(-u\)-direction, one has \( \varepsilon(u) = \varepsilon(0) \exp(pu) \) where \( p \) is pertinently called the taper parameter. Eq.(4) simplifies to \( U'' + pU' - n^2 U = 0 \), and the solutions for the coefficients \( \{b_n\}_n^{\infty} \) (cf. (6)) become

\[
b_0 = \frac{p \exp(pu_n)}{1 - \exp(pu_n)}, \quad b_n = \frac{r_{1n} \exp(-r_{1n} u_n) - r_{2n} \exp(-r_{2n} u_n)}{\exp(-r_{1n} u_n) - \exp(-r_{2n} u_n)}, \quad n = 1, 2, \ldots
\]

(13)

where \( r_{(1,2)n} = \frac{1}{2} p \pm \sqrt{p^2 + 4n^2} \).
Asymptotic solutions. For large values of \( n \), and with certain restrictions on the parameters \( \varepsilon(u) \) and \( \mu(u) \), approximate solutions to the radial equation (4) can be found by analysis similar to the asymptotic techniques described in [2].

If the conditions \(|\varepsilon'/\varepsilon| \ll n\) and \(|\varepsilon''/\varepsilon'| \ll n\) are fulfilled the solution for \( b_n \) is approximately

\[
b_n \approx -\left( n + \alpha'(0)/n \right) \frac{\varepsilon'(0)}{\varepsilon(0)} \coth \left( n u_s - \alpha(0)/n \right) - \frac{1}{2} \frac{\varepsilon'(0)}{\varepsilon(0)}.
\]

(14)

where \( \alpha(u) = \frac{1}{4} \left( \frac{\varepsilon'(u)}{\varepsilon(u)} - \frac{\varepsilon'(u_s)}{\varepsilon(u_s)} \right) + \frac{1}{8} \int_{u_s}^{u} \left( \varepsilon'(t) \right)^2 dt. \)

The benefit in cases when the asymptotic expression (14) can be used is that instead of doing \( N_v + 1 \) numerical integrations of equation (4), it suffices with a much smaller number of integrations and (at most) one additional numerical integration for determining \( \alpha(0) \).

Numerical Example

Consider the geometry in figure 1, with the shielding at \( u_s = 0.5 \), \( a = 50 \) mm, and three planar conductors at \(-35 < x/a < -15, 0 < x/a < 10 \) and \( 35 < x/a < 50 \), respectively. The medium parameters are: \( \varepsilon_r(u) = \mu_r(u) = 10 \exp(-u^2 \ln 10) \), i.e. the tapers are \( p_q \approx -4.6, p_{(1/\mu)} \approx 4.6 \); cf. eq.(13).

The phase velocities for the propagating modes, computed using the scheme in [1], are presented in table 1 below, for different truncation numbers \( N_v \), for the basis functions, and for different values of \( N_a \), which is the lowest value of \( n \) for which the radial equation has been solved asymptotically.

<table>
<thead>
<tr>
<th>( N_a )</th>
<th>( &gt; N_v )</th>
<th>10</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Phase velocity for mode 1 (10^7 m/s)</td>
<td>( N_v = 200 )</td>
<td>3.176</td>
<td>3.176</td>
<td>3.172</td>
<td>3.165</td>
</tr>
<tr>
<td></td>
<td>( N_v = 400 )</td>
<td>3.174</td>
<td>3.174</td>
<td>3.169</td>
<td>3.162</td>
</tr>
<tr>
<td>(b) Phase velocity for mode 2 (10^7 m/s)</td>
<td>( N_v = 200 )</td>
<td>4.907</td>
<td>4.907</td>
<td>4.883</td>
<td>4.826</td>
</tr>
<tr>
<td></td>
<td>( N_v = 400 )</td>
<td>4.912</td>
<td>4.912</td>
<td>4.890</td>
<td>4.832</td>
</tr>
<tr>
<td>(c) Phase velocity for mode 3 (10^7 m/s)</td>
<td>( N_v = 200 )</td>
<td>7.149</td>
<td>7.149</td>
<td>7.139</td>
<td>7.031</td>
</tr>
<tr>
<td></td>
<td>( N_v = 400 )</td>
<td>7.151</td>
<td>7.151</td>
<td>7.141</td>
<td>7.034</td>
</tr>
</tbody>
</table>

Table 1: The phase velocities for the three propagation modes, for the geometry in figure 1, with exponential parameters \( \varepsilon_r(u) = \mu_r(u) = 10 \exp(-u^2 \ln 10) \). The exact expression (13) for \( b_n \) has been used when \( 0 \leq n < N_a \); the asymptotic expression (14) has been used when \( N_a \leq n \leq N_v \).

In table 1, one sees that the results obtained using the asymptotic expression for \( b_n \) are very accurate even down to \( N_a = 3 \). The reason is that the exponential profile is well suited for asymptotic analysis, since the conditions \(|\varepsilon'/\varepsilon| \ll n\) and \(|\varepsilon''/\varepsilon'| \ll n\) are fulfilled when the taper parameter satisfies \(|p| \ll n\).

For the exponential profile, the expressions (13) and (14) are in fact identical up to the second order in a Taylor series expansion in \( p \).

References
