An Analytical Method of Auxiliary Sources Solution for Plane Wave Scattering by Impedance Cylinders

Niels Vesterdal Larsen and Olav Breindbjerg,
Orsted DTU, Electromagnetic Systems,
Technical University of Denmark,
Building 348, Orsteds Plads, DK-2800 Kgs. Lyngby, Denmark,
Tel: +45 4528 3800, Fax: +45 4593 1634, E-mail: ob@oersted.dtu.dk

Abstract: Analytical Method of Auxiliary Sources solutions for plane wave scattering by circular impedance cylinders are derived by transformation of the exact eigenfunction series solutions employing the Hankel function wave transformation. The analytical Method of Auxiliary Sources solution thus obtained serves as a reference for numerical implementations of the Method of Auxiliary Sources and verifies the accuracy of the latter.

INTRODUCTION

The Method of Auxiliary Sources (MAS), see Kaklamani et al.[1], is a numerical technique which, for an exterior scattering problem, can be explained as follows. The discrete auxiliary sources (AS) are positioned on an auxiliary surface located inside the scatterer. They are chosen as Hertzian dipoles or line currents whose radiated fields are given by closed form expressions. Letting the AS radiate in free space, the excitations of these are then determined from point matching of the boundary condition on the surface of the scatterer and by solving the linear system of equations thus established.

Several investigations have been reported, see Anastassiu et al.[2] and Karamehmedovic et al.[3,4], on the applicability of MAS to scattering problems where the scatterer satisfies the Standard Impedance Boundary Condition (SIBC) by examining the boundary condition error and far fields. It is, however, also of relevance to investigate whether the excitation coefficients of the AS are calculated appropriately.

The purpose of this work is to derive an analytical MAS solution, and then use this as a reference for the numerical MAS. This is done for transverse magnetic (TM) and transverse electric (TE) plane wave illumination of circular SIBC cylinders.

The analytical MAS solution is derived by a transformation of the exact eigenfunction series solution employing the addition theorem for cylindrical functions which is also often referred to as the Hankel function wave transformation. For a given choice of the auxiliary surface and the number N of AS, this analytical MAS solution is optimal in the sense that it recovers accurately the eigenfunctions series solution truncated to N terms. The reader is referred to Larsen et al.[5] for the details of this work.

NUMERICAL MAS FORMULATION AND EXACT SOLUTION

The scattering configuration, see figure 1, consists of an infinitely long circular impedance cylinder, with the surface impedance \( Z_s \), located in free space with intrinsic impedance \( Z_0 \). The cylinder is illuminated by a time harmonic plane wave \( (\vec{E}, \vec{H}) \), with angular frequency \( \omega \), and wave number \( k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \). The plane wave may be TM- or TE-polarized with respect to the cylinder axis. The outward unit normal vector to the scatterer surface is denoted \( \vec{n} \). The cylinder radius is \( r_0 \) and the surface is \( S \). The time dependence \( e^{j \omega t} \) is assumed and suppressed throughout the text. A rectangular \((x, y, z)\)-coordinate system and an associated circular cylindrical \((r, \phi, z)\)-coordinate system, are employed.

The SIBC requires that the total field \( \{ \vec{E}, \vec{H} \} \), being the sum of the incident field \( (\vec{E}^i, \vec{H}^i) \) and the scattered field \( (\vec{E}^s, \vec{H}^s) \), satisfies the relation \( \vec{n} \times \vec{E} = Z_s \vec{n} \times (\vec{n} \times \vec{H}) \). The exact eigenfunction series solution of the homogeneous Helmholtz’s wave equation, can be found by employing separation of variables. With \( D_k \) being constants depending on \( Z_s, r_0 \), and the polarization (see Larsen et al.[5]) and \( H^{(2)}_k \) being Hankel functions of the second kind and order \( k \), this solution can be expressed as

\[
\begin{aligned}
\{ \vec{E}^{i,\text{TM}}, \vec{H}^{i,\text{TE}} \} &= -j \omega \{ \vec{E}^i \} \left\{ \frac{1}{Z_0} \right\} \sum_{k=-\infty}^{\infty} j^k D_k H^{(2)}_k(k_0 r) e^{j k \phi}.
\end{aligned}
\]

The AS, distributed on the auxiliary surface \( S_a \) of radius \( r_a \) conformal to \( S \), are chosen as infinitely long electric or magnetic line sources oriented parallel to the cylinder axis. The AS radiate an electric or
Auxiliary Sources (AS)

Test points

\[ \mathbf{\hat{n}} \times \mathbf{\hat{E}} - Z_n \mathbf{\hat{n}} \times (\mathbf{\hat{n}} \times \mathbf{\hat{H}}) \]

**Figure 1: Scattering configuration and MAS setup.**

- Magnetic field, which is proportional to \( H_0^{(2)}(k_0 r) \) corresponding to the TM or TE configuration, respectively.
- By applying \( M = N \) test points on \( S \) a quadratic system of equations is formed and the excitation coefficients of the AS are calculated to yield the MAS solution

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathbf{\hat{E}}_{\text{mas}}^{+, \text{TM}} \\
Z_n \mathbf{\hat{H}}_{\text{mas}}^{+, \text{TE}}
\end{array} \right\} = \hat{z} \sum_{n=1}^{N} C_n^{\text{MAS}} H_0^{(2)}(k_0 |\mathbf{\hat{r}} - \mathbf{\hat{r}}_n|),
\end{align*}
\]

where \( \mathbf{\hat{r}}_n \) and \( C_n^{\text{MAS}} \) describe the position and excitation of the \( n \)’th AS.

- The results included here are for a TM configuration with \( Z_n = 100 \, \Omega \) and a TE configuration with \( Z_n = (100 + 100j) \, \Omega \). The physical surface of the cylinder is \( r_0 = 2\lambda \), where \( \lambda \) is the wavelength, and the auxiliary surface radius \( r_a = 1.5\lambda \). The results are shown in figures 2 and 3.

**ANALYTICAL MAS FORMULATION**

The analytical MAS is based on a transformation of the eigenfunction series solution (1) into a MAS solution. This is accomplished by taking outset in the MAS solution and - employing a wave transformation for the Hankel function - rewriting this in terms of the the eigenfunction series solution. Presently, a truncated version of the wave transform is used and the MAS solution (2) is thus approximated as

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathbf{\hat{E}}_{\text{mas}}^{+, \text{TM}} \\
Z_n \mathbf{\hat{H}}_{\text{mas}}^{+, \text{TE}}
\end{array} \right\} = \hat{z} \sum_{n=1}^{N} C_n^{\text{MAS}} H_0^{(2)}(k_0 |\mathbf{\hat{r}} - \mathbf{\hat{r}}_n|) \approx \hat{z} \sum_{n=1}^{N} \left[ C_n^{\text{MAS}} \sum_{k=-K}^{K} J_k(k_0 r_n) H_0^{(2)}(k_0 r) e^{jk(\phi - \phi_n)} \right],
\end{align*}
\]

where \( r_n \) and \( \phi_n \) describe the positions of the AS. An interchange of the summations on the right hand side of (3) allows a direct identification with the eigenfunction series solution (1) and it is evident that the MAS solution (3) approximately recovers the eigenfunction series solution (1) truncated to \( N = 2K + 1 \) terms provided that

\[
-|\mathbf{\hat{E}}|^2 D_k = \sum_{n=1}^{N} C_n^{\text{MAS}} J_k(k_0 r_n) e^{-j(k_0 r_n)}, \quad k \in [-K, K].
\]

The error introduced by using the truncated wave transformation can be shown to be very small. The quadratic system of equations may in general be solved as such, however, a less general choice of equidistant AS with \( \phi_n = \frac{2\pi n}{2K} \) and fixed \( r_n = r_a \) corresponding to a concentric auxiliary surface, allows a more elegant solution by means of a discrete Fourier transformation.

The analytical MAS is optimal in the sense that it recovers accurately the truncated exact eigenfunction solution and thus yield similarly optimal excitation coefficients. The amplitude of the excitation coefficients found using the two different MAS approaches are exemplified in figure 2. In general the excitations of the analytical MAS are recovered by the numerical MAS to a very high degree. Further investigations, not included here, have
shown that the small discrepancies become even smaller as the number of AS is increased. The corresponding far fields are shown in figure 3. Again the numerical MAS recovers the analytical MAS solution very well.

**SUMMARY AND CONCLUSIONS**

An analytical reference MAS for plane wave incidence on circular impedance cylinders was derived. A comparison between the analytical and numerical MAS showed good agreement regarding the excitations of the auxiliary sources and far fields. In conclusion, it has been shown that the numerical MAS yields results that are in good agreement with the optimal, analytical MAS results as far as excitation coefficients and especially far fields are concerned.

**REFERENCES**


