Measure-Transformed Two-Sample Hotelling Test: Supplementary Material

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In Section I of this supplementary material document we provide proofs for Proposition 1 and Theorem 1. A proof for relation (25) is given in Section II.

I. Proof of Proposition 1 and Theorem 1

First, we derive the asymptotic distribution of the scaled difference $\sqrt{\frac{N'M'}{N'+M'}}\left(\hat{\boldsymbol{\theta}}_1^{(u_1)}-\hat{\boldsymbol{\theta}}_2^{(u_2)}\right)$ under the null hypothesis and under the local alternatives in (17). Using Eqs. (5) and (9)-(12) one can verify that the difference

$$\hat{\boldsymbol{\theta}}_1^{(u_1)}-\hat{\boldsymbol{\theta}}_2^{(u_2)}=\mathbf{S}_{\mathbf{A}}\left(\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{W}}}^{(u_1)}-\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{Z}}}^{(u_2)}\right)+\left(\boldsymbol{\theta}_1-\boldsymbol{\theta}_2\right),\qquad \text{(S-1)}$$

where $\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{W}}}^{(u_1)}$ and $\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{Z}}}^{(u_2)}$ denote the empirical MT-mean vectors of $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{Z}}$, respectively, and $\mathbf{S}_{\mathbf{A}}$ is defined below Eq. (12). In Lemma 1, stated below, we show that under condition (A-2)

$$\sqrt{N'}\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{W}}}^{(u_1)}\xrightarrow[N'\to\infty]{D}\mathcal{N}\left(\mathbf{0},\Gamma_{\tilde{\mathbf{W}}}^{(u_1)}\boldsymbol{\Sigma}_{\tilde{\mathbf{W}}}^{(u_1^2)}\right)\qquad \text{(S-2)}$$

and

$$\sqrt{M'}\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{Z}}}^{(u_2)}\xrightarrow[M'\to\infty]{D}\mathcal{N}\left(\mathbf{0},\Gamma_{\tilde{\mathbf{Z}}}^{(u_2)}\boldsymbol{\Sigma}_{\tilde{\mathbf{W}}}^{(u_2^2)}\right),\qquad \text{(S-3)}$$

where

$$\Gamma_{\tilde{\mathbf{W}}}^{(u_1)}\triangleq\mathrm{E}[\varphi_{u_1}^2(\tilde{\mathbf{W}});P_{\tilde{\mathbf{W}}}],\quad \Gamma_{\tilde{\mathbf{Z}}}^{(u_2)}\triangleq\mathrm{E}[\varphi_{u_2}^2(\tilde{\mathbf{Z}});P_{\tilde{\mathbf{Z}}}]\qquad \text{(S-4)}$$

and $\varphi_u(\cdot)$ is defined below Eq. (1). Under the null hypothesis $\boldsymbol{\theta}_1-\boldsymbol{\theta}_2=\mathbf{0}$, whereas under the local alternatives in (17) and Assumption (A-1)

$$\sqrt{\frac{N'M'}{N'+M'}}(\boldsymbol{\theta}_1-\boldsymbol{\theta}_2)\xrightarrow[N',M'\to\infty]{}\sqrt{\frac{c}{1+c}}\mathbf{h}_1-\sqrt{\frac{1}{1+c}}\mathbf{h}_2\triangleq\mathbf{h}.\qquad \text{(S-5)}$$

Thus, by (S-1)-(S-5), Assumption (A-1), the property that $\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{W}}}^{(u_1)}$ and $\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{Z}}}^{(u_2)}$ are uncorrelated, and Slutskey's Theorem [s1], it follows that under the null hypothesis

$$\sqrt{\frac{N'M'}{N'+M'}}\left(\hat{\boldsymbol{\theta}}_1^{(u_1)}-\hat{\boldsymbol{\theta}}_2^{(u_2)}\right)\xrightarrow[N',M'\to\infty]{D}\mathcal{N}\left(\mathbf{0},\mathbf{B}_{u_1,u_2}\right)\qquad \text{(S-6)}$$

and under the local alternatives in (17)

$$\sqrt{\frac{N'M'}{N'+M'}}\left(\hat{\boldsymbol{\theta}}_1^{(u_1)}-\hat{\boldsymbol{\theta}}_2^{(u_2)}\right)\xrightarrow[N',M'\to\infty]{D}\mathcal{N}\left(\mathbf{h},\mathbf{B}_{u_1,u_2}\right),\qquad \text{(S-7)}$$

where

$$\mathbf{B}_{u_1,u_2}\triangleq\frac{c}{1+c}\mathbf{R}_{\tilde{\mathbf{W}}}^{(u_1)}+\frac{1}{1+c}\mathbf{R}_{\tilde{\mathbf{Z}}}^{(u_2)},\qquad \text{(S-8)}$$

$$\mathbf{R}_{\tilde{\mathbf{W}}}^{(u_1)}\triangleq\Gamma_{\tilde{\mathbf{W}}}^{(u_1)}\mathbf{S}_{\mathbf{A}}\boldsymbol{\Sigma}_{\tilde{\mathbf{W}}}^{(u_1^2)}\mathbf{S}_{\mathbf{A}}^T\quad\text{and}\quad\mathbf{R}_{\tilde{\mathbf{Z}}}^{(u_2)}\triangleq\Gamma_{\tilde{\mathbf{Z}}}^{(u_2)}\mathbf{S}_{\mathbf{A}}\boldsymbol{\Sigma}_{\tilde{\mathbf{Z}}}^{(u_2^2)}\mathbf{S}_{\mathbf{A}}^T.\qquad \text{(S-9)}$$

Next, we prove that $\hat{\mathbf{B}}_{u_1,u_2}$ in (14) is a strongly consistent estimator of $\mathbf{B}_{u_1,u_2}$ in (S-8). By Eqs. (5), (6), (9) and (10), it follows that $\hat{\mathbf{B}}_{u_1,u_2}$ can be written as:

$$\hat{\mathbf{B}}_{u_1,u_2}=\frac{M'}{M'+N'}\hat{\mathbf{R}}_{\tilde{\mathbf{W}}}^{(u_1)}+\frac{N'}{M'+N'}\hat{\mathbf{R}}_{\tilde{\mathbf{Z}}}^{(u_2)},\qquad \text{(S-10)}$$

where

$$\hat{\mathbf{R}}_{\tilde{\mathbf{W}}}^{(u_1)}\triangleq\hat{\Gamma}_{\tilde{\mathbf{W}}}^{(u_1)}\mathbf{S}_{\mathbf{A}}\hat{\boldsymbol{\Sigma}}_{\tilde{\mathbf{W}}}^{(u_1^2)}\mathbf{S}_{\mathbf{A}}^T,\quad \hat{\mathbf{R}}_{\tilde{\mathbf{Z}}}^{(u_2)}\triangleq\hat{\Gamma}_{\tilde{\mathbf{Z}}}^{(u_2)}\mathbf{S}_{\mathbf{A}}\hat{\boldsymbol{\Sigma}}_{\tilde{\mathbf{Z}}}^{(u_2^2)}\mathbf{S}_{\mathbf{A}}^T,\qquad \text{(S-11)}$$

$$\hat{\Gamma}_{\tilde{\mathbf{W}}}^{(u_1)}\triangleq N'\sum_{n=1}^{N'}\hat{\varphi}_{u_1}^2(\tilde{\mathbf{W}}_n)\quad\text{and}\quad\hat{\Gamma}_{\tilde{\mathbf{Z}}}^{(u_2)}\triangleq M'\sum_{m=1}^{M'}\hat{\varphi}_{u_2}^2(\tilde{\mathbf{Z}}_m).\qquad \text{(S-12)}$$

In Lemma 2 below, we show that when Assumption (A-2) is satisfied

$$\hat{\mathbf{R}}_{\tilde{\mathbf{W}}}^{(u_1)}\xrightarrow[N'\to\infty]{\text{w.p.1}}\mathbf{R}_{\tilde{\mathbf{W}}}^{(u_1)}\quad\text{and}\quad\hat{\mathbf{R}}_{\tilde{\mathbf{Z}}}^{(u_2)}\xrightarrow[M'\to\infty]{\text{w.p.1}}\mathbf{R}_{\tilde{\mathbf{Z}}}^{(u_2)}.\qquad \text{(S-13)}$$

Therefore, by (S-8), (S-10), (S-13), Assumption (A-1) and Mann-Wald's Theorem [s2] we conclude that

$$\hat{\mathbf{B}}_{u_1,u_2}\xrightarrow[N',M'\to\infty]{\text{w.p.1}}\mathbf{B}_{u_1,u_2}.\qquad \text{(S-14)}$$

We now show that $\mathbf{B}_{u_1,u_2}$ in (S-8) is exactly the one reported in (20). Using Eqs. (3), (4), (9), (10) and (S-4), it can be shown that

$$\Gamma_{\tilde{\mathbf{W}}}^{(u_1)}=\Gamma_{\tilde{\mathbf{X}}}^{(u_1)},\qquad\Gamma_{\tilde{\mathbf{Z}}}^{(u_2)}=\Gamma_{\tilde{\mathbf{Y}}}^{(u_2)},\qquad \text{(S-15)}$$

$$\boldsymbol{\Sigma}_{\tilde{\mathbf{W}}}^{(u_1^2)}=\boldsymbol{\Sigma}_{\tilde{\mathbf{X}}}^{(u_1^2)}\quad\text{and}\quad\boldsymbol{\Sigma}_{\tilde{\mathbf{Z}}}^{(u_2^2)}=\boldsymbol{\Sigma}_{\tilde{\mathbf{Y}}}^{(u_2^2)}.\qquad \text{(S-16)}$$

Therefore, by (S-9) we have

$$\mathbf{R}_{\tilde{\mathbf{W}}}^{(u_1)}=\mathbf{R}_{\tilde{\mathbf{X}}}^{(u_1)}\quad\text{and}\quad\mathbf{R}_{\tilde{\mathbf{Z}}}^{(u_2)}=\mathbf{R}_{\tilde{\mathbf{Y}}}^{(u_2)},\qquad \text{(S-17)}$$

which implies that (S-8) takes the form:

$$\mathbf{B}_{u_1,u_2}=\frac{c}{1+c}\mathbf{R}_{\tilde{\mathbf{X}}}^{(u_1)}+\frac{1}{1+c}\mathbf{R}_{\tilde{\mathbf{Y}}}^{(u_2)}.\qquad \text{(S-18)}$$

Finally, the relations in (16) and (18) follow from (13), (S-6), (S-7), (S-14), (S-18), Slutskey's Theorem [s1], Mann-Wald's Theorem [s2] and the properties of quadratic forms of Gaussian random variables [s3]. □

Lemma 1. *Relations (S-2) and (S-3) hold under Assumption (A-2).*

Proof: We only prove relation (S-2). Proof of (S-3) is similar and therefore omitted. By Eq. (5), the empirical MT-mean of $\tilde{\mathbf{W}}$ takes the forms:

$$\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{W}}}^{(u_1)}=\frac{\frac{1}{N'}\sum_{n=1}^{N'}u_1(\tilde{\mathbf{W}}_n)\tilde{\mathbf{W}}_n}{\frac{1}{N'}\sum_{n=1}^{N'}u_1(\tilde{\mathbf{W}}_n)}=g(\mathbf{T}),\qquad \text{(S-19)}$$

where $\mathbf{T} \triangleq [\mathbf{T}_1^T, T_2]^T$, $\mathbf{T}_1 \triangleq \frac{1}{N'}\sum_{n=1}^{N'} u_1(\tilde{\mathbf{W}}_n)\tilde{\mathbf{W}}_n$, $T_2 \triangleq \frac{1}{N'}\sum_{n=1}^{N'} u_1(\tilde{\mathbf{W}}_n)$ and $g(\mathbf{T}) \triangleq \frac{\mathbf{T}_1}{T_2}$. Furthermore, by Eq. (3), the MT-mean of $\tilde{\mathbf{W}}$ is given by:

$$\boldsymbol{\mu}_{\tilde{\mathbf{W}}}^{(u_1)} = \frac{\mathrm{E}[u_1(\tilde{\mathbf{W}})\tilde{\mathbf{W}}; P_{\tilde{\mathbf{W}}}]}{\mathrm{E}[u_1(\tilde{\mathbf{W}}); P_{\tilde{\mathbf{W}}}]} = g(\mathrm{E}[\mathbf{T}; P_{\mathbf{T}}]) = \mathbf{0}. \tag{S-20}$$

The last equality in (S-20) follows from the properties that $\tilde{\mathbf{W}}$ is symmetrically distributed about the origin and the MT-function $u_1(\cdot)$ is zero-centered and symmetric. Recall that $\{\tilde{\mathbf{W}}_n\}_{n=1}^{N'}$ is a sequence of i.i.d. samples from $P_{\tilde{\mathbf{W}}}$. Therefore, since by Assumption (A-2)

$$\mathrm{E}[u_1^2(\tilde{\mathbf{W}}); P_{\tilde{\mathbf{W}}}] < \infty \text{ and } \mathrm{E}[u_1^2(\tilde{\mathbf{W}})\|\tilde{\mathbf{W}}\|^2; P_{\tilde{\mathbf{W}}}] < \infty, \tag{S-21}$$

it follows from the central limit theorem [s4, Th. 11.2.4] that

$$\sqrt{N'}\left(\mathbf{T} - \mathrm{E}[\mathbf{T}; P_{\mathbf{T}}]\right) \xrightarrow[N'\to\infty]{D} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{V}}\right), \tag{S-22}$$

where $\boldsymbol{\Sigma}_{\mathbf{V}} \triangleq \mathrm{cov}[\mathbf{V}; P_{\mathbf{V}}]$ and $\mathbf{V} \triangleq u_1(\tilde{\mathbf{W}})[\tilde{\mathbf{W}}^T, 1]^T$. Now, by Eqs. (S-19), (S-20), (S-22) and the delta-method [s4, Th. 11.2.14], we conclude that

$$\sqrt{N'}\hat{\boldsymbol{\mu}}_{\tilde{\mathbf{W}}}^{(u_1)} \xrightarrow[N'\to\infty]{D} \mathcal{N}\left(\mathbf{0}, \mathbf{G}\boldsymbol{\Sigma}_{\mathbf{V}}\mathbf{G}^T\right), \tag{S-23}$$

where the gradient matrix $\mathbf{G} \triangleq \left.\frac{dg(\mathbf{T})}{d\mathbf{T}}\right|_{\mathbf{T}=\mathrm{E}[\mathbf{T}; P_{\mathbf{T}}]}$. Using Eqs. (3), (4), (S-4), (S-19) and (S-20), one can verify that

$$\mathbf{G}\boldsymbol{\Sigma}_{\mathbf{V}}\mathbf{G}^T = \Gamma_{\tilde{\mathbf{W}}}^{(u_1)}\boldsymbol{\Sigma}_{\tilde{\mathbf{W}}}^{(u_1^2)}, \tag{S-24}$$

which completes the proof. ∎

Lemma 2. *The convergences in (S-13) holds under Assumption (A-2).*

Proof: In the following, we prove strong consistency of $\hat{\mathbf{R}}_{\tilde{\mathbf{W}}}^{(u_1)}$. Consistency proof of $\hat{\mathbf{R}}_{\tilde{\mathbf{Z}}}^{(u_2)}$ is similar and therefore omitted. By (S-11), $\hat{\mathbf{R}}_{\tilde{\mathbf{W}}}^{(u_1)} \triangleq \hat{\Gamma}_{\tilde{\mathbf{W}}}^{(u_1)}\mathbf{S}_{\mathbf{A}}\hat{\boldsymbol{\Sigma}}_{\tilde{\mathbf{W}}}^{(u_1^2)}\mathbf{S}_{\mathbf{A}}^T$, where $\hat{\Gamma}_{\tilde{\mathbf{W}}}^{(u_1)}$ is defined in (S-12). By Assumption (A-2), the expectation $\mathrm{E}[u_1^2(\tilde{\mathbf{W}})\|\tilde{\mathbf{W}}\|^2; P_{\tilde{\mathbf{W}}}]$ is finite. Therefore, since $\{\tilde{\mathbf{W}}_n\}_{n=1}^{N'}$ is a sequence of i.i.d. samples from $P_{\tilde{\mathbf{W}}}$, it follows from [s5, Prop. 2] that

$$\hat{\boldsymbol{\Sigma}}_{\tilde{\mathbf{W}}}^{(u_1^2)} \xrightarrow[N'\to\infty]{\text{w.p.1}} \boldsymbol{\Sigma}_{\tilde{\mathbf{W}}}^{(u_1^2)}. \tag{S-25}$$

By Assumption (A-2) we further have that $\mathrm{E}[u_1^2(\tilde{\mathbf{W}}); P_{\tilde{\mathbf{W}}}] < \infty$. Thus, by (S-12), Khinchine's strong law of large numbers (KSLLN) [s6] and Mann-Wald's Theorem [s2] it follows that

$$\begin{aligned}\hat{\Gamma}_{\tilde{\mathbf{W}}}^{(u_1)} &= \frac{\frac{1}{N'}\sum_{n=1}^{N'} u_1^2(\tilde{\mathbf{W}}_n)}{\left(\frac{1}{N'}\sum_{n=1}^{N'} u_1(\tilde{\mathbf{W}}_n)\right)^2} \\ &\xrightarrow[N'\to\infty]{\text{w.p.1}} \frac{\mathrm{E}[u_1^2(\tilde{\mathbf{W}}); P_{\tilde{\mathbf{W}}}]}{\mathrm{E}^2[u_1(\tilde{\mathbf{W}}); P_{\tilde{\mathbf{W}}}]} = \Gamma_{\tilde{\mathbf{W}}}^{(u_1)}.\end{aligned} \tag{S-26}$$

The last equality in (S-26) follows from (S-4) and the definition of the function $\varphi_u(\cdot)$ below Eq. (1). To conclude, by (S-11), (S-25), (S-26) and Mann-Wald's Theorem [s2] we have that $\hat{\mathbf{R}}_{\tilde{\mathbf{W}}}^{(u_1)} \xrightarrow[N'\to\infty]{\text{w.p.1}} \mathbf{R}_{\tilde{\mathbf{W}}}^{(u_1)}$. ∎

II. Proof of Relation (25)

Notice that by Eq. (20),

$$\mathrm{tr}[\mathbf{B}_{u_1,u_2}] = \frac{c}{1+c}\mathrm{tr}\left[\mathbf{R}_{\tilde{\mathbf{X}}}^{(u_1)}\right] + \frac{1}{1+c}\mathrm{tr}\left[\mathbf{R}_{\tilde{\mathbf{Y}}}^{(u_2)}\right]. \tag{S-27}$$

In the following, we shall obtain an expression for $\mathrm{tr}[\mathbf{R}_{\tilde{\mathbf{X}}}^{(u_1)}]$. Using the definition of $\mathbf{R}_{\tilde{\mathbf{X}}}^{(u_1)}$, below (20), and Eqs. (4) and (24), one can verify that

$$\begin{aligned}\mathrm{tr}\left[\mathbf{R}_{\tilde{\mathbf{X}}}^{(u_1)}\right] &= \frac{\mathrm{E}\left[u_1^2(\tilde{\mathbf{X}})\left\|\mathbf{S}_{\mathbf{A}}\left(\tilde{\mathbf{X}} - \boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1^2)}\right)\right\|^2; P_{\tilde{\mathbf{X}}}\right]}{\mathrm{E}^2\left[u_1(\tilde{\mathbf{X}}); P_{\tilde{\mathbf{X}}}\right]} \\ &= \frac{\mathbf{a}_1^T\mathbf{C}_{\tilde{\mathbf{X}}}\mathbf{a}_1}{\mathbf{a}_1^T\mathbf{v}_{\tilde{\mathbf{X}}}\mathbf{v}_{\tilde{\mathbf{X}}}^T\mathbf{a}_1},\end{aligned} \tag{S-28}$$

where

$$\mathbf{v}_{\tilde{\mathbf{X}}} \triangleq \mathrm{E}[\boldsymbol{\psi}(\mathbf{P}_{\mathbf{A}}^{\perp}\tilde{\mathbf{X}}); P_{\tilde{\mathbf{X}}}] \tag{S-29}$$

and

$$\mathbf{C}_{\tilde{\mathbf{X}}} \triangleq \mathrm{E}[\boldsymbol{\psi}(\mathbf{P}_{\mathbf{A}}^{\perp}\tilde{\mathbf{X}})\boldsymbol{\psi}^T(\mathbf{P}_{\mathbf{A}}^{\perp}\tilde{\mathbf{X}})\|\mathbf{S}_{\mathbf{A}}\mathbf{d}_{\tilde{\mathbf{X}}}\|^2; P_{\tilde{\mathbf{X}}}]. \tag{S-30}$$

The vector

$$\mathbf{d}_{\tilde{\mathbf{X}}} \triangleq \tilde{\mathbf{X}} - \boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1^2)}. \tag{S-31}$$

We now find an expression for the MT-mean $\boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1^2)}$ in (S-31). Using Eqs. (3) and (24), one can verify that

$$\boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1)} = \mathbf{D}_{\tilde{\mathbf{X}}}\mathbf{a}_1/\left(\mathbf{v}_{\tilde{\mathbf{X}}}^T\mathbf{a}_1\right), \tag{S-32}$$

where $\mathbf{D}_{\tilde{\mathbf{X}}} \triangleq \mathrm{E}[\tilde{\mathbf{X}}\boldsymbol{\psi}^T(\mathbf{P}_{\mathbf{A}}^{\perp}\tilde{\mathbf{X}}); P_{\tilde{\mathbf{X}}}]$. Notice that according to Eq. (11), $\boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1)}$ is invariant over the class of MT-functions satisfying (10), i.e., the class $\mathcal{V} \triangleq \left\{v(\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{r}), \mathbf{r} \in \mathbb{R}^{2p}\right\}$ of non-negative zero-centered symmetric functions. Also note that if $u_1(\cdot)$ belongs to $\mathcal{V}$, then also $u_1^2(\cdot)$. Therefore, by (24) we conclude that

$$\boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1)} = \boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1^2)} = \mathbf{A}\boldsymbol{\theta}_1$$

for *any* coefficients vector $\mathbf{a}_1 \in \mathbb{R}_+^K$. Hence, by setting $\mathbf{a}_1 = \mathbf{1}$, we obtain

$$\boldsymbol{\mu}_{\tilde{\mathbf{X}}}^{(u_1^2)} = \mathbf{D}_{\tilde{\mathbf{X}}}\mathbf{1}/\left(\mathbf{v}_{\tilde{\mathbf{X}}}^T\mathbf{1}\right). \tag{S-33}$$

Therefore, by substituting (S-33) into (S-31) we obtain the same vector $\mathbf{d}_{\tilde{\mathbf{X}}}$ that appears below Eq. (27) in the paper.

Using the same arguments above, it can be shown that

$$\mathrm{tr}\left[\mathbf{R}_{\tilde{\mathbf{Y}}}^{(u_2)}\right] = \frac{\mathbf{a}_2^T\mathbf{C}_{\tilde{\mathbf{Y}}}\mathbf{a}_2}{\mathbf{a}_2^T\mathbf{v}_{\tilde{\mathbf{Y}}}\mathbf{v}_{\tilde{\mathbf{Y}}}^T\mathbf{a}_2}, \tag{S-34}$$

with $\mathbf{v}_{\tilde{\mathbf{Y}}}$ and $\mathbf{C}_{\tilde{\mathbf{Y}}}$ defined in Eqs. (26) and (27), respectively. Relation (25) follows directly from (S-27), (S-28) and (S-34). □

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