A Note on the Measure-Transformed Multiple Signal Classification Algorithm

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***Abstract*—In this paper, we extend the class of measure transformation functions under which the measure-transformed MUSIC algorithm can be implemented. We prove that the noise subspace can be determined from the eigen-decomposition of the measure-transformed covariance when the measure transformation function belongs to the wide class of strictly-positive spherically contoured functions.**

***Index Terms*—Array processing, DOA estimation, probability measure transform, robust estimation, signal subspace estimation.**

I. Introduction

Recently, we developed in [1] a robust MUSIC generalization, called measure-transformed MUSIC (MT-MUSIC) that is based on applying a transform to the probability measure of the data. The considered probability measure transformation is structured by a non-negative function, called MT-function, that weights the data points. In MT-MUSIC, the sample covariance matrix is replaced by an empirical estimate of the measure-transformed (MT) covariance. The MT-MUSIC algorithm was confined to the family of spherically contoured *Gaussian* MT-functions. Under this class of MT-functions it was shown that the empirical MT-covariance is B-robust [2]. Under the additional assumption of spherically symmetric compound Gaussian (CG) noise [3], we proved the key property that the vectors spanning the noise subspace can be determined from the eigen-decomposition of the MT-covariance.

A natural question that immediately arises is that whether the MT-MUSIC algorithm can be implemented with other types of MT-functions that do not necessarily belong to the *Gaussian* family. In this paper we provide an answer. Under the assumption of spherically symmetric CG noise (also made in [1]), we prove that when the MT-function belongs to the wider class of strictly-positive spherically contoured functions, the spanning vectors of the noise subspace can be determined via eigen-decomposition of the MT-covariance. This result paves the way for development of other *non-Gaussian* MT-MUSIC algorithms that have different useful properties compared to the *Gaussian* MT-MUSIC considered in [1].

II. Implementation of the MT-MUSIC algorithm with spherically contoured MT-functions

We first note that all required background regarding the considered array model, probability measure transformation, and the MT-MUSIC algorithm appears in detail in [1, Sec. II]. In the following, we assume that the MT-function [1, Def. 1] is spherically contoured that takes the form:

$$u\left(\mathbf{x}\right)=h\left(\|\mathbf{x}\|\right), \tag{1}$$

where $h:\mathbb{R}_{+}\rightarrow\mathbb{R}_{++}$ is a strictly-positive function. Clearly, by (1) and [1, Prop. 3] the B-robustness condition [1, Cond. 1, Sec. II-E], is satisfied whenever there exists a positive constant M such that

$$h(t)\leq M \text{ and } t^{2}h(t)\leq M \text{ for any } t\in\mathbb{R}_{+}. \tag{2}$$

Furthermore, we assume that the noise component in the array model [1, Eq. (1)] is a spherically symmetric CG random vector that satisfies the following stochastic representation [3]:

$$\mathbf{W}_{n}=\nu_{n}\mathbf{Z}_{n}, \tag{3}$$

where $\{\nu_{n}\in\mathbb{R}_{++}\}$ is a first-order stationary process and $\{\mathbf{Z}_{n}\in\mathbb{C}^{p}\}$ is a proper-complex wide-sense stationary Gaussian process with zero-mean and scaled unit covariance $\sigma^{2}_{\mathbf{Z}}\mathbf{I}$. The processes $\{\nu_{n}\}$ and $\{\mathbf{Z}_{n}\}$ are assumed to be statistically independent.

In the following Theorem, the structure of the resulting MT-covariance [1, Eq. (8)] is derived. This structure provides a non-trivial extension to the one stated in [1, Th. 1] for the specific case of spherically contoured Gaussian MT-function. For simplicity, we shall assume that the MT-mean [1, Eq. (9)] is zero. Under the MT-function (1) and the symmetrically distributed noise (3) this assumption is satisfied when the signal component $\mathbf{S}$ in the array model [1, Eq. (1)] is symmetrically distributed about the origin.

Theorem 1. *Under the observation model [1, Eq. (1)], the MT-function (1) and the CG noise (3), the MT-covariance [1, Eq. (8)] takes the form:*

$$\mathbf{\Sigma}^{(u)}_{\mathbf{X}}=\mathbf{A}\mathbf{J}^{(u)}\mathbf{A}^{H}+\alpha^{(u)}\mathbf{I}_{p}, \tag{4}$$

where

$$\mathbf{J}^{(u)}\triangleq \mathrm{E}\left[\mathbf{S}\mathbf{S}^{H}\beta_{p}\left(\kappa\right)\frac{\|\mathbf{X}\|^{4}}{\nu^{4}\sigma^{4}_{\mathbf{Z}}}\varphi_{h}\left(\|\mathbf{X}\|\right);P_{\mathbf{X},\mathbf{S},\nu}\right]$$

is a strictly positive-definite matrix,

$$\alpha^{(u)}\triangleq \mathrm{E}\left[\gamma_{p}\left(\kappa\right)\|\mathbf{X}\|^{2}\varphi_{h}\left(\|\mathbf{X}\|\right);P_{\mathbf{X},\mathbf{S},\nu}\right],$$

$\mathbf{I}_{p}$ *is a* $p\times p$ *identity matrix,* $\kappa\triangleq 2\|\mathbf{X}\|\|\mathbf{A}\mathbf{S}\|/(\nu^{2}\sigma^{2}_{\mathbf{Z}})$*,*

$$\beta_{p}\left(\kappa\right)\triangleq\begin{cases}\frac{4I_{p+1}(\kappa)}{\kappa^{2}I_{p-1}(\kappa)}, & \text{if } \kappa>0\\ 0, & \text{if } \kappa=0\end{cases},$$

$$\gamma_{p}\left(\kappa\right)\triangleq\begin{cases}\frac{2I_{p}(\kappa)}{\kappa I_{p-1}(\kappa)}, & \text{if } \kappa>0\\ 1/p, & \text{if } \kappa=0\end{cases},$$

$I_{k}(\cdot)$ *is the k-th order modified Bessel function of the first kind [4], and* $\varphi_{h}\left(\|\mathbf{X}\|\right)\triangleq h(\|\mathbf{X}\|)/\mathrm{E}[h(\|\mathbf{X}\|);P_{\mathbf{X}}]$*.*

The structure (4), along with the property that the matrix $\mathbf{J}^{(u)}$ is non-singular and the assumption that steering matrix $\mathbf{A}$ in the array model [1, Eq. (1)] has a full column rank, imply that Condition 2 stated in [1, Sec. II-E] is satisfied. Under this key condition, the spanning vectors of the null-space of $\mathbf{A}^{H}$, also called the noise subspace, can be determined from the eigen-decomposition of the MT-covariance. By modifying the MT-function in the wide class defined by (1) a family of MT-MUSIC algorithms can be obtained. In particular, when $h(\cdot)$ is any non-zero constant function, the standard non-robust MUSIC algorithm is obtained. Selection of $h(t)=\exp(-t^{2}/\omega^{2})$, $\omega\in\mathbb{R}_{++}$, results in the robust Gaussian MT-MUSIC considered in [1]. Other robust MT-MUSIC procedures can be obtained by selecting different MT-functions that satisfy (1) and (2), e.g., $h(t)=1/(1+\exp(t/\omega))$, $t\in\mathbb{R}_{+}$.

Proof of Theorem 1. By the definition of the MT-covariance [1, Eq. (8)], relation (1), and the assumption of zero MT-mean, the MT-covariance takes the form:

$$\begin{aligned}\mathbf{\Sigma}^{(u)}_{\mathbf{X}} &= \mathrm{E}[\mathbf{X}\mathbf{X}^{H}\varphi_{h}(\|\mathbf{X}\|);P_{\mathbf{X}}] \\ &= \mathrm{E}\left[\mathrm{E}\left[\mathbf{X}\mathbf{X}^{H};P_{\mathbf{X}|\mathbf{S},\nu,\|\mathbf{X}\|}\right]\varphi_{h}(\|\mathbf{X}\|);P_{\mathbf{S},\nu,\|\mathbf{X}\|}\right],\end{aligned} \tag{5}$$

where $\varphi_{h}(\cdot)$ is defined below (4), and $P_{\cdot|\cdot}$ denotes a conditional probability measure. The second equality in (5) follows directly from the law of total expectation. Notice that by [1, Eq. (1)] and (3) the random vector $\mathbf{X}|\mathbf{S},\nu\sim\mathcal{CN}\left(\mathbf{A}\mathbf{S},\nu^{2}\sigma^{2}_{\mathbf{Z}}\mathbf{I}_{p}\right)$.Thus, by Lemma 1 stated below, we obtain that the conditional expectation

$$\mathrm{E}\left[\mathbf{X}\mathbf{X}^{H};P_{\mathbf{X}|\mathbf{S},\nu,\|\mathbf{X}\|}\right]=\beta_{p}\left(\kappa\right)\frac{\|\mathbf{X}\|^{4}}{\nu^{4}\sigma^{4}_{\mathbf{Z}}}\mathbf{A}\mathbf{S}\mathbf{S}^{H}\mathbf{A}^{H}+\gamma_{p}\left(\kappa\right)\|\mathbf{X}\|^{2}\mathbf{I}_{p}, \tag{6}$$

where $\beta_p(\cdot)$, $\gamma_p(\cdot)$ and κ are defined below (4). Finally, relation (4) is obtained by substituting (6) into (5).

We now prove that the non-negative matrix $\mathbf{J}^{(u)}$ defined below (4) is strictly positive-definite, i.e., non-singular. We show that if $\mathbf{J}^{(u)}$ is singular, then the signal covariance matrix $\boldsymbol{\Sigma}_{\mathbf{S}} \triangleq \mathrm{E}\left[\mathbf{S}\mathbf{S}^H; P_{\mathbf{S}}\right]$, which under the array model [1, Eq. (1)] is assumed to be non-singular, must be singular. Under the assumption that $\mathbf{J}^{(u)}$ is singular, there exists a non-zero vector $\mathbf{v} \in \mathbb{C}^p$ such that

$$\mathbf{v}^H\mathbf{J}^{(u)}\mathbf{v} = \mathrm{E}\left[\left|\mathbf{v}^H\mathbf{S}\right|^2\beta_p\left(\kappa\right)\frac{\|\mathbf{X}\|^4}{\nu^4\sigma_{\mathbf{Z}}^4}\varphi_h\left(\|\mathbf{X}\|\right); P_{\mathbf{X},\mathbf{S},\kappa}\right] = 0. \quad (7)$$

Notice that by [1, Eq. (1)] and the CG noise (3) the observation vector $\mathbf{X}$ is a continuous random vector, and therefore, $\|\mathbf{X}\| > 0$ w.p. 1. Also note that $\varphi_h(\cdot)$, ν and $\sigma_{\mathbf{Z}}^4$ are strictly positive. Hence, by (7) and proposition 2.3.9 in [5], we conclude that $|\mathbf{v}^H\mathbf{S}|^2\beta_p\left(\kappa\right) = 0$ w.p. 1. In the following, we show that the latter equality implies that $|\mathbf{v}^H\mathbf{S}|^2 = 0$ w.p. 1. Clearly, if $\beta_p\left(\kappa\right) > 0$ then $|\mathbf{v}^H\mathbf{S}|^2 = 0$. Now, if $\beta_p\left(\kappa\right) = 0$, it follows from the definition of $\beta_p\left(\kappa\right)$ that $\kappa = 0$. Therefore, by the definition of κ, below (4), we conclude that since $\|\mathbf{X}\|$ and ν are strictly positive w.p. 1 and $\sigma_{\mathbf{Z}}^4 > 0$, then $\|\mathbf{A}\mathbf{S}\| = 0$ w.p. 1. Thus, since the steering matrix $\mathbf{A}$ has a full column rank, we conclude that when $\kappa = 0$ the signal vector $\mathbf{S} = \mathbf{0}$ w.p. 1. Therefore,

$$\mathbf{v}^H\boldsymbol{\Sigma}_{\mathbf{S}}\mathbf{v} = \mathrm{E}\left[\left|\mathbf{v}^H\mathbf{S}\right|^2; P_{\mathbf{S}}\right] = 0,$$

which implies that $\boldsymbol{\Sigma}_{\mathbf{S}}$ is singular. □

Lemma 1. *Let* $\mathbf{Q} \sim \mathcal{CN}\left(\boldsymbol{\mu}, \sigma^2\mathbf{I}_p\right)$, *where* $\mathbf{Q} \in \mathbb{C}^p$. *Then*

$$\mathrm{E}\left[\mathbf{Q}\mathbf{Q}^H; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right] = \beta_p\left(\kappa\right)\frac{\|\mathbf{Q}\|^4}{\sigma^4}\boldsymbol{\mu}\boldsymbol{\mu}^H + \|\mathbf{Q}\|^2\gamma_p\left(\kappa\right)\mathbf{I}_p,$$

where $\beta_p(\cdot)$ *and* $\gamma_p(\cdot)$ *are defined below (4) and* $\kappa \triangleq 2\|\mathbf{Q}\|\|\boldsymbol{\mu}\|/\sigma^2$.

Proof. First, notice that since $\mathbf{Q}$ is a continuous random vector, then $\|\mathbf{Q}\| > 0$ w.p. 1. Therefore, $\kappa = 0$ w.p. 1 if and only if $\boldsymbol{\mu} = \mathbf{0}$. The proof begins for the special case of $\boldsymbol{\mu} = \mathbf{0}$, which is equivalent to $\kappa = 0$ w.p. 1. Note that in this case, the random vector $\mathbf{Q}$ is spherically symmetric, and therefore, $\mathbf{Q} \overset{d}{=} \mathbf{U}\mathbf{Q}$ for any unitary matrix $\mathbf{U} \in \mathbb{C}^{p\times p}$ [3], where "$\overset{d}{=}$" denotes equality in distribution. Hence, the equality $\mathrm{E}\left[\mathbf{Q}\mathbf{Q}^H; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right] = \mathbf{U}\mathrm{E}\left[\mathbf{Q}\mathbf{Q}^H; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right]\mathbf{U}^H$ holds for any unitary matrix $\mathbf{U} \in \mathbb{C}^{p\times p}$, which implies that

$$\mathrm{E}\left[\mathbf{Q}\mathbf{Q}^H; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right] = \frac{\mathrm{tr}\left\{\mathrm{E}\left[\mathbf{Q}\mathbf{Q}^H; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right]\right\}}{p}\mathbf{I}_p = \frac{\|\mathbf{Q}\|^2}{p}\mathbf{I}_p, \quad (8)$$

where $\mathrm{tr}\{\cdot\}$ denotes the trace operator.

Now, we assume that $\boldsymbol{\mu} \neq \mathbf{0}$, which is equivalent to $\kappa > 0$ w.p. 1. Here, we shall use the relation

$$\mathrm{E}\left[\mathbf{Q}\mathbf{Q}^H; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right] = \mathrm{E}\left[\frac{\mathbf{Q}\mathbf{Q}^H}{\|\mathbf{Q}\|^2}; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right]\|\mathbf{Q}\|^2 \quad (9)$$

and derive the conditional expectation in the r.h.s. of (9). Let $\mathbf{T}_1 \triangleq \mathrm{Re}\{\mathbf{Q}\}$, $\mathbf{T}_2 \triangleq \mathrm{Im}\{\mathbf{Q}\}$ and $\mathbf{T} \triangleq [\mathbf{T}_1^T, \mathbf{T}_2^T]^T$. Notice that

$$\mathrm{E}\left[\frac{\mathbf{T}\mathbf{T}^T}{\|\mathbf{T}\|^2}; P_{\mathbf{T}|\|\mathbf{T}\|=\rho}\right] = \mathrm{E}\left[\mathbf{T}'\mathbf{T}'^T; P_{\mathbf{T}'|\|\mathbf{T}'\|=1}\right] \triangleq \mathbf{D}, \quad (10)$$

where $\mathbf{T}' \triangleq \mathbf{T}/\rho \sim \mathcal{N}(c\cdot\boldsymbol{\eta}, c\cdot\kappa^{-1}\mathbf{I})$, $\boldsymbol{\eta} \triangleq \boldsymbol{\mu}_{\mathbf{T}}/\|\boldsymbol{\mu}_{\mathbf{T}}\|$, $\boldsymbol{\mu}_{\mathbf{T}} \triangleq [\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T]^T$, $\boldsymbol{\mu}_1 \triangleq \mathrm{Re}\{\boldsymbol{\mu}\}$, $\boldsymbol{\mu}_2 \triangleq \mathrm{Im}\{\boldsymbol{\mu}\}$, $\kappa \triangleq 2\rho\|\boldsymbol{\mu}_{\mathbf{T}}\|/\sigma^2$, and $c \triangleq \|\boldsymbol{\mu}_{\mathbf{T}}\|/\rho$. By [6, p. 379], it follows that the conditional distribution of $\mathbf{T}'$ given that $\|\mathbf{T}'\| = 1$ is $M_{2p}(\boldsymbol{\eta}, \kappa)$, where $M_{2p}(\boldsymbol{\eta}, \kappa)$ is the Von-Mises Fisher distribution [7] on the $2p-1$ dimensional unit sphere $\mathbb{S}_{2p-1}$, with directional mean $\boldsymbol{\eta}$ and concentration parameter κ. Hence, the density function of the conditional probability measure $P_{\mathbf{T}'|\|\mathbf{T}'\|=1}$ is given by [7]:

$$f_{\mathbf{T}'|\|\mathbf{T}'\|=1}\left(\boldsymbol{t}\right) \triangleq C\left(\kappa\right)\exp\left(\kappa\boldsymbol{\eta}^T\boldsymbol{t}\right), \quad \boldsymbol{t} \in \mathbb{S}_{2p-1}, \quad (11)$$

where $C(\kappa) \triangleq \kappa^{p-1}/((2\pi)^p I_{p-1}(\kappa))$ is a normalization constant. Define the vector $\tilde{\boldsymbol{\eta}} \triangleq \kappa\boldsymbol{\eta}$. Notice that $\kappa = \|\tilde{\boldsymbol{\eta}}\|$ and thus

$$\frac{d\kappa}{d\tilde{\boldsymbol{\eta}}} = \frac{\tilde{\boldsymbol{\eta}}}{\kappa}. \quad (12)$$

Furthermore, by the recursive relation 10.51.6 in [8] one can verify that

$$\left(\frac{1}{\kappa}\frac{d}{d\kappa}\right)^m C^{-1}\left(\kappa\right) = C^{-1}\left(\kappa\right)\frac{I_{p-1+m}\left(\kappa\right)}{\kappa^m I_{p-1}\left(\kappa\right)}. \quad (13)$$

Therefore, by (11)−(13) and Theorem 2.40 in [9] for differentiation under the integral sign, we conclude that the conditional autocorrelation matrix in (10) takes the form:

$$\begin{aligned}\mathbf{D} &= C(\kappa)\frac{d^2}{d\tilde{\boldsymbol{\eta}}d\tilde{\boldsymbol{\eta}}^T}\int_{\mathbb{S}_{2p-1}}\exp\left(\tilde{\boldsymbol{\eta}}^T\boldsymbol{t}\right)d\omega_{2p-1}(\boldsymbol{t}) \quad (14)\\ &= C(\kappa)\frac{d^2}{d\tilde{\boldsymbol{\eta}}d\tilde{\boldsymbol{\eta}}^T}C^{-1}(\kappa)\\ &= C(\kappa)\frac{d\kappa}{d\tilde{\boldsymbol{\eta}}}\frac{d}{d\kappa}\left(\frac{d\kappa}{d\tilde{\boldsymbol{\eta}}}\frac{d}{d\kappa}C^{-1}(\kappa)\right)^T\\ &= C(\kappa)\left(\left(\frac{1}{\kappa}\frac{d}{d\kappa}\right)C^{-1}(\kappa)\mathbf{I}+\tilde{\boldsymbol{\eta}}\tilde{\boldsymbol{\eta}}^T\left(\frac{1}{\kappa}\frac{d}{d\kappa}\right)^2C^{-1}(\kappa)\right)\\ &= \frac{I_p(\kappa)}{\kappa I_{p-1}(\kappa)}\mathbf{I}+\tilde{\boldsymbol{\eta}}\tilde{\boldsymbol{\eta}}^T\frac{I_{p+1}(\kappa)}{\kappa^2 I_{p-1}(\kappa)}\\ &= \frac{I_p(\kappa)}{\kappa I_{p-1}(\kappa)}\mathbf{I}+\boldsymbol{\eta}\boldsymbol{\eta}^T\frac{I_{p+1}(\kappa)}{I_{p-1}(\kappa)},\end{aligned}$$

where ω_{2p-1} denotes Lebegue's surface measure on $\mathbb{S}_{2p-1}$. Therefore, by (10) and (14)

$$\mathrm{E}\left[\frac{\mathbf{T}_k\mathbf{T}_j^T}{\|\mathbf{T}\|^2}; P_{\mathbf{T}|\|\mathbf{T}\|=\rho}\right] = \frac{I_p(\kappa)}{\kappa I_{p-1}(\kappa)}\mathbf{I}\delta_{k,j} + \frac{\boldsymbol{\mu}_k\boldsymbol{\mu}_j^T}{\|\boldsymbol{\mu}_{\mathbf{T}}\|^2}\frac{I_{p+1}(\kappa)}{I_{p-1}(\kappa)}, \quad (15)$$

k,j=1,2, where $\delta_{k,j}$ is the Kronecker delta function. Finally, by (15) and the definition of $\mathbf{T}$, we conclude that

$$\begin{aligned}&\mathrm{E}\left[\frac{\mathbf{Q}\mathbf{Q}^H}{\|\mathbf{Q}\|^2}; P_{\mathbf{Q}|\|\mathbf{Q}\|}\right] \quad (16)\\ &= \mathrm{E}\left[\frac{\mathbf{T}_1\mathbf{T}_1^T}{\|\mathbf{T}\|^2}; P_{\mathbf{T}|\|\mathbf{T}\|=\rho}\right] + i\mathrm{E}\left[\frac{\mathbf{T}_2\mathbf{T}_1^T}{\|\mathbf{T}\|^2}; P_{\mathbf{T}|\|\mathbf{T}\|=\rho}\right]\\ &\quad -i\mathrm{E}\left[\frac{\mathbf{T}_1\mathbf{T}_2^T}{\|\mathbf{T}\|^2}; P_{\mathbf{T}|\|\mathbf{T}\|=\rho}\right] + \mathrm{E}\left[\frac{\mathbf{T}_2\mathbf{T}_2^T}{\|\mathbf{T}\|^2}; P_{\mathbf{T}|\|\mathbf{T}\|=\rho}\right]\\ &= \frac{2I_p(\kappa)}{\kappa I_{p-1}(\kappa)}\mathbf{I} + \frac{I_{p+1}(\kappa)}{I_{p-1}(\kappa)}\frac{\boldsymbol{\mu}\boldsymbol{\mu}^H}{\|\boldsymbol{\mu}\|^2}\\ &= \frac{2I_p(\kappa)}{\kappa I_{p-1}(\kappa)}\mathbf{I} + \frac{4I_{p+1}(\kappa)}{\kappa^2 I_{p-1}(\kappa)}\frac{\|\mathbf{Q}\|^2}{\sigma^4}\boldsymbol{\mu}\boldsymbol{\mu}^H.\end{aligned}$$

The proof of the Lemma follows directly from (8), (9) and (16). □

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